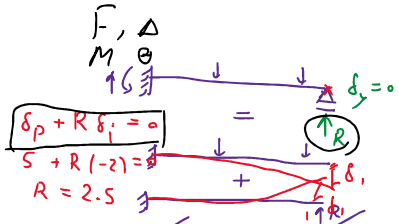
Introduction

Sunday, October 8, 2023 8:42

تحلیل سازه ها I

سرفصل درس



① پایداری و معین سازه ها

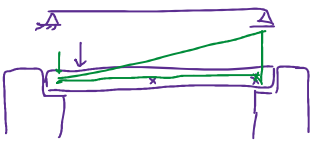
② تحلیل سازه های معین استاتیکی

③ محاسبه تغییر شکل های سازه به روش های کلی سطح، بار الاستیک و تیر مزدوج

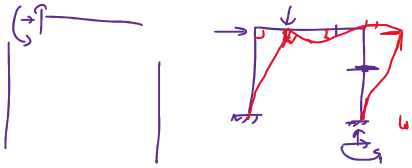
④ محاسبه تغییر شکل ها به روش های انرژی شامل کار مجازی و بار واحد، قضایای کاستیلیانو و مضرب بتی و ماکسول

⑤ تحلیل سازه های نامعین به روش نیرویی (انرژی): روش سازگاری تغییر شکل ها

⑥ خط تأثیر سازه های معین

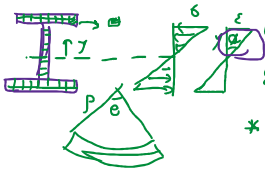


موضوع درس تحلیل سازه ها:



F, Δ

بارهای وارده ← شامل نیروها و تغییر شکل ها



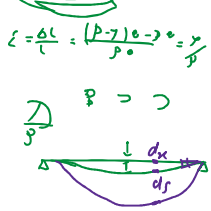
مقاومت مصالح: الان میگرد ε, σ

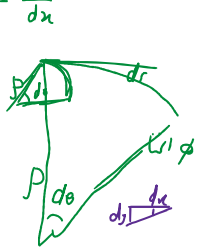
تحلیل سازه ها: الان ماگرد F, Δ

$\sigma = \frac{My}{I}$

$\varepsilon = \frac{y}{\rho} \rightarrow \phi = \frac{1}{\rho} = \frac{\varepsilon}{y} = \frac{\sigma}{Ey} = \frac{My}{EI}$

$\phi = \frac{d\theta}{dx}$





$ds = \rho\, d\theta$

$\rho = \frac{ds}{d\theta}$

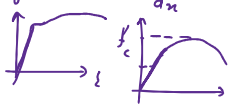
$\phi = \frac{1}{\rho} = \frac{d\theta}{ds}$

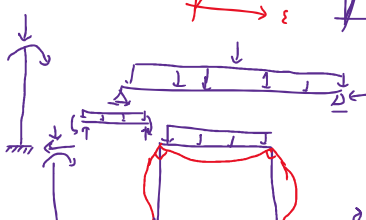
$e = \frac{dy}{dx} \rightarrow y = \int \theta\, dx + C_2$

$\phi = \frac{\varepsilon}{y} = \frac{\sigma}{Ey} = \frac{My}{EIy} = \frac{M}{EI}$

$\phi = \frac{d\theta}{dx}$

$\frac{d\theta}{dx} = \frac{M}{EI} \rightarrow \theta = \int \frac{M}{EI} dx + C_1$



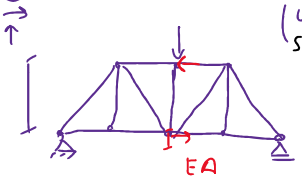


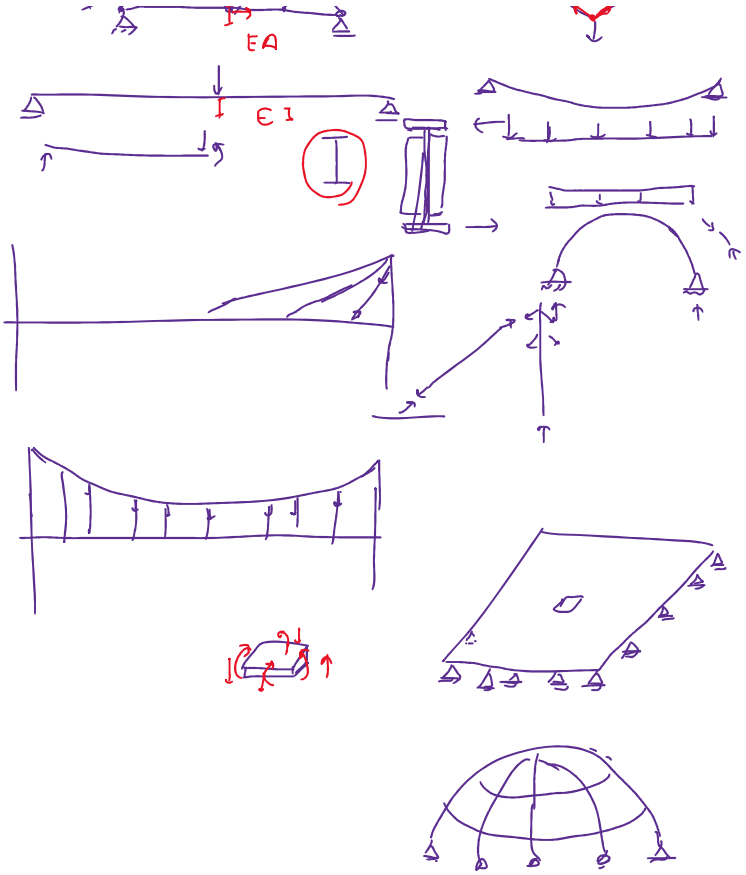
انواع سازه: ① تیر، ستون و تاب Frame Column Beam

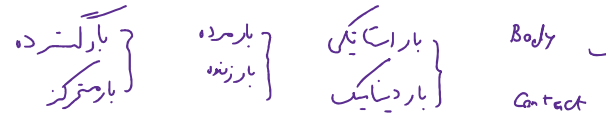
② خرپاها Truss

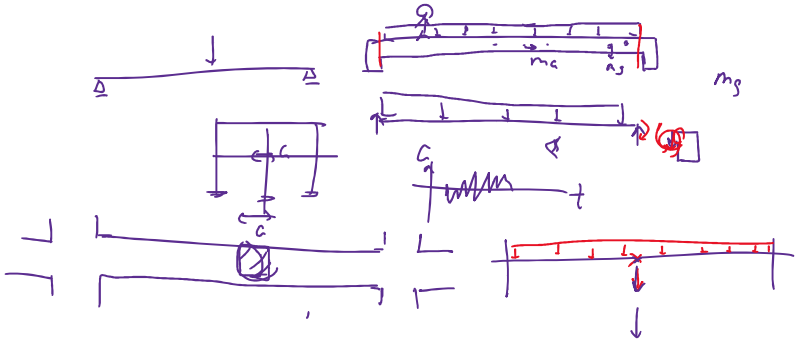
③ کابل ها و قوس ها Arches Cables

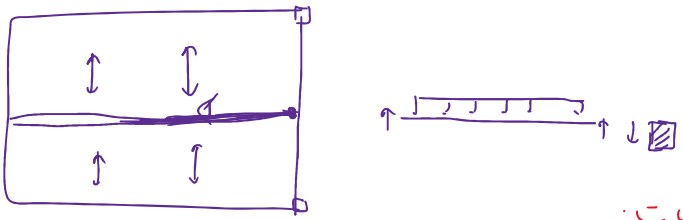
④ سازه های سطحی (صفحه ها و پوسته ها) Shell Plate Surface structures





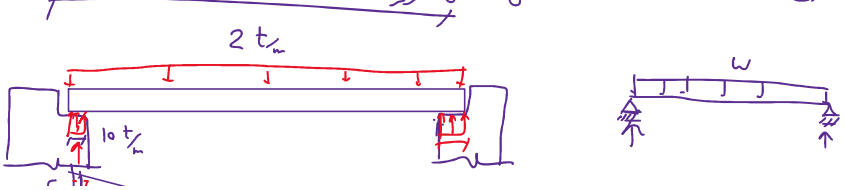


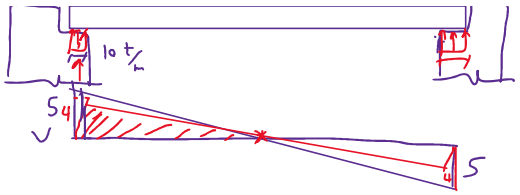


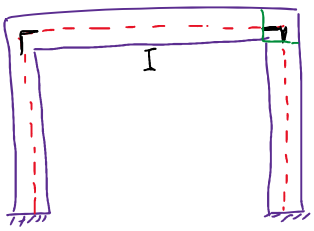


فرضیات:

① فرض بارهای متمرکز و عکس العمل تکیه گاهی متمرکز







۲) المان ها را به صورت خط Centerline

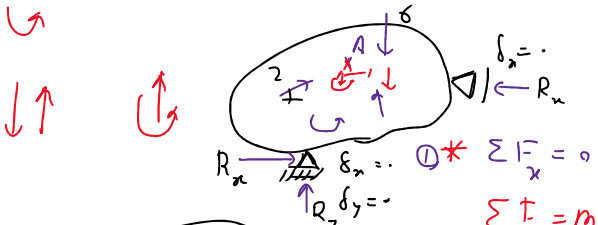
۳) رفتار خطی مصالح $\sigma = E\varepsilon$

$\frac{1}{\rho} = \frac{M}{EI}$

۱) صرف نظر از تغییر شکل ها (فرض تغییر شکل های کوچک)

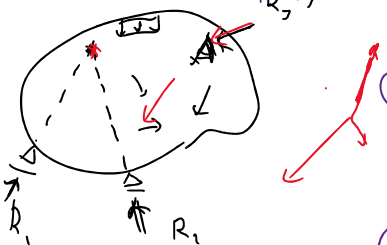
فصل اول:

پایداری و معین سازه ها



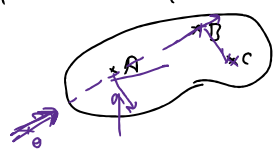
* ① $\Sigma F_x = 0$ $\Sigma F_y = 0$ $\boxed{\Sigma M_A = 0}$

$\Sigma F_x = ma_x$ $\Sigma F_y = ma_y$ $\Sigma M = I\alpha$



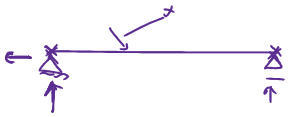
② $\Sigma M_A = 0$ $\Sigma M_B = 0$ $\Sigma F_y = 0$

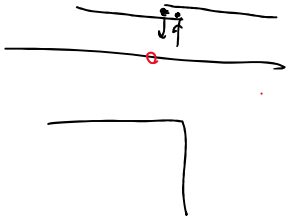
به شرطی که محور معادله نیرو بر امتداد AB عمود نباشد.

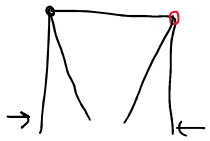


③ $\Sigma M_A = 0$ $\Sigma M_B = 0$ $\Sigma M_C = 0$

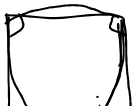
به شرطی که A، B و C در یک امتداد نباشند.

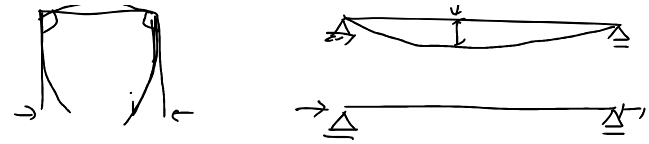






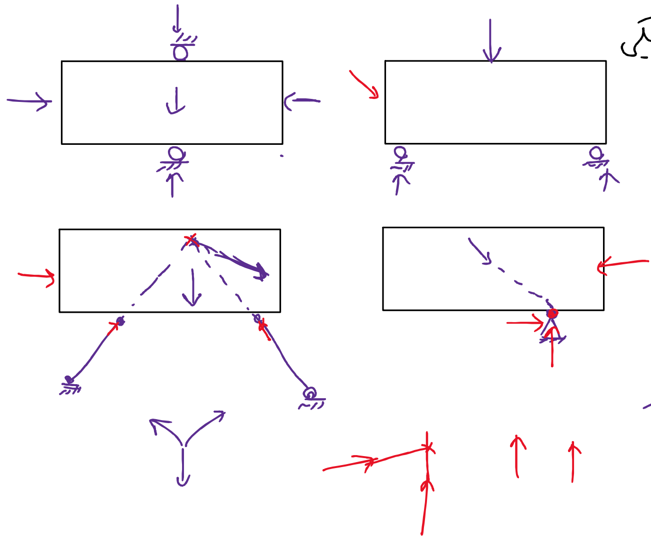
{ Rigid body motion حرکت
{ deformation تغییر شکل

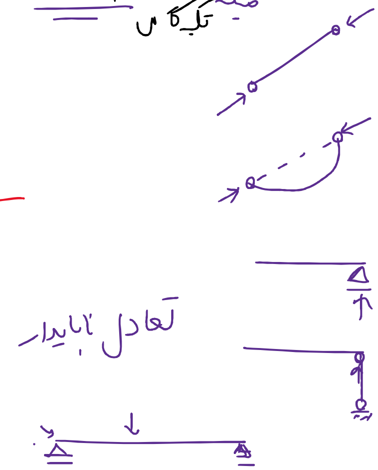




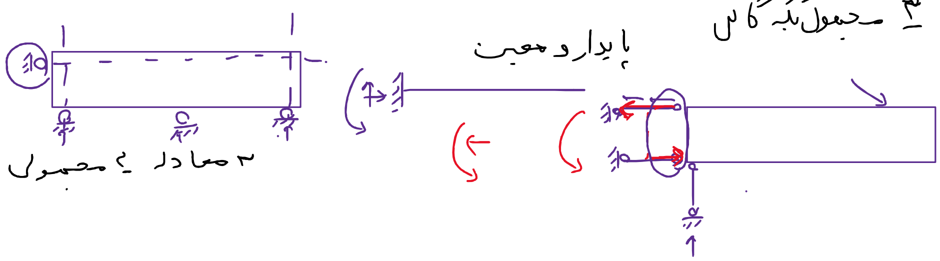
پایداری و معینی یک جسم صلب

① کمتر از ۳ مجهول ناپایدار (استاتیکی)
صلب
تکیه گاهی

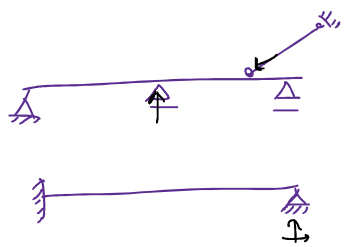




② ۳ مجهول تکیه گاهی



③ بیش از ۳ مجهول تکیه گاهی



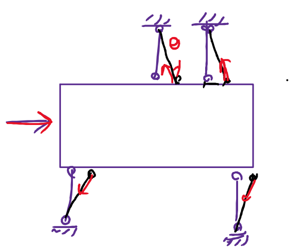
$5 - 3 = 2$

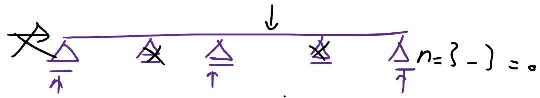
پایدار و نامعین

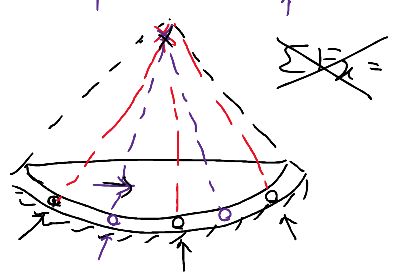
2

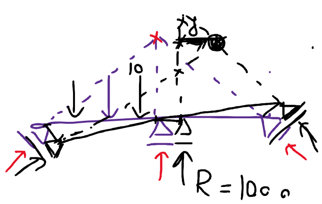
④ ۳ مجهول تکیه گاهی لازم است ولی کافی نیست

ناپایداری هندسی خارجی

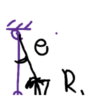






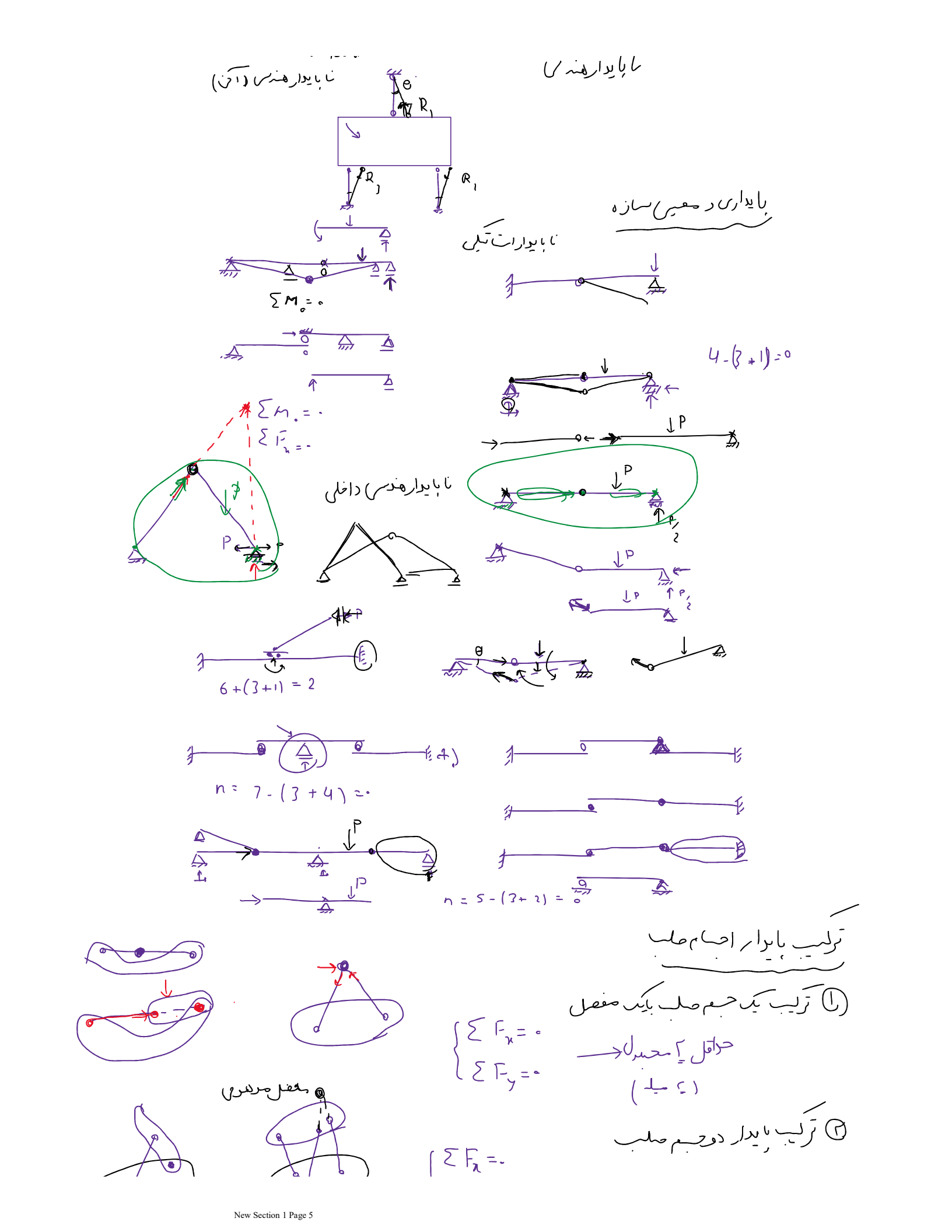


$n = 3 - 3 = 0$

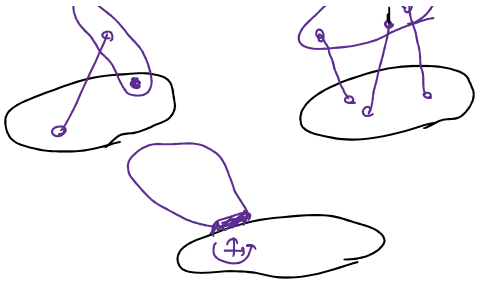


ناپایدار هندسی (آنی)

ناپایدار هندسی

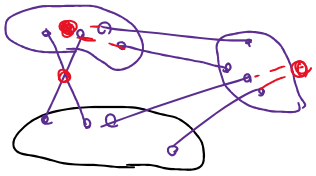

ناپایدار هندسی (آنی)
ناپایداری هندسی
R_1
R_1
R_1
$\sum M_o = 0$
پایداری و معین سازه
ناپایداری تکی
$4-(3+1)=0$
$\sum M_o = 0$
$\sum F_x = 0$
P
ناپایداری هندسی داخلی
$6+(3+1)=2$
$n = 7-(3+4)=0$
$n = 5-(3+2)=0$
ترکیب پایدار اجسام صلب
(۱) ترکیب یک جسم صلب با یک مفصل
حداقل ۲ مجهول
(۲ میله)
$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \end{cases}$
(۲) ترکیب پایدار دو جسم صلب
مفصل مرتبه
$\sum F_x = 0$

② ترکیب پایدار دو جسم صلب



$\Sigma F_x = 0$
$\Sigma F_y = 0$ → حداقل ۳ مجهول
$\Sigma M = 0$

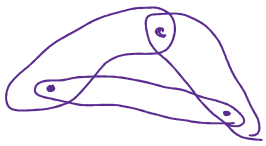
③ ترکیب پایدار سه جسم صلب



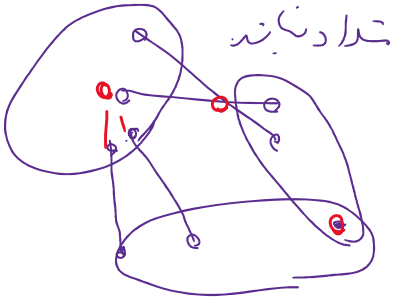
۶ مجهول → ۶ معادله

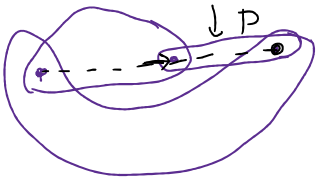
① ۳ مفصل نه در یک استقامت نباشند

② ۶ میله که سر در جسم صلب با ۳ میله وصل شود و مفاصل سر دیگر یک امتداد نباشند.

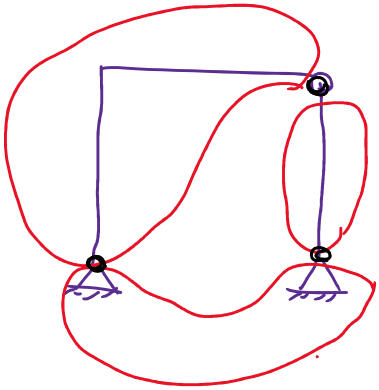


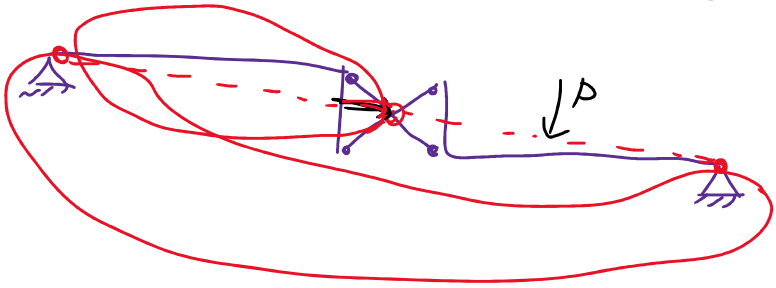
③ ترکیبی از مفصل و میله که مفاصل سر دیگر و حقیقی در یک امتداد نباشند



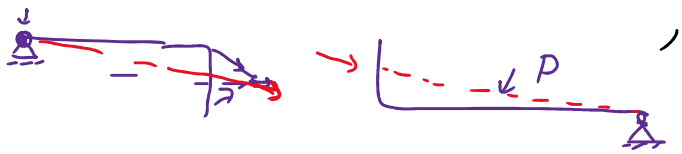


مثال:

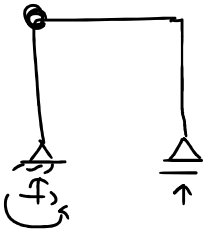




$n = 0$ پایدار



ناپایدار



$n = 4 - 3 = 1$

درجه نامعینی سازه:

تعداد معادلات − تعداد مجهولات = n

مجهولات تکیه گاهی

تعداد مجهولات تکیه گاهی $|n|$

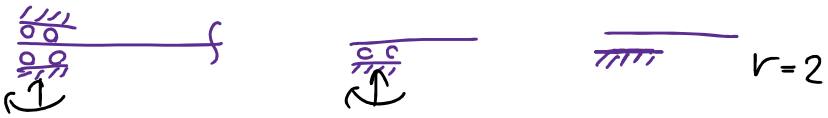
تعداد مجهولات تکیه گاهی (r)

① غلتکی

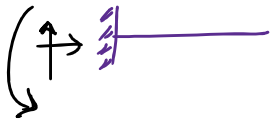
② مفصلی

③ مفصل برش

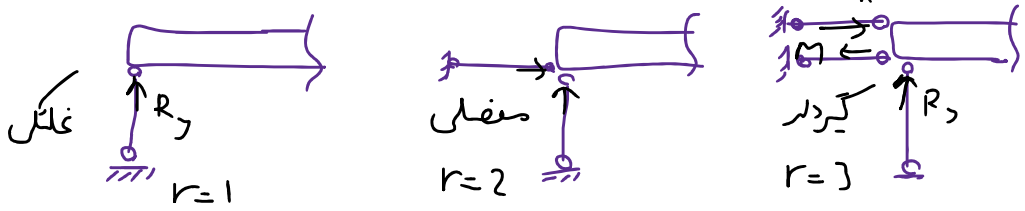
④ مفصل یزدی محدود

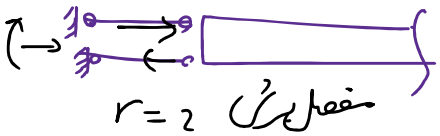


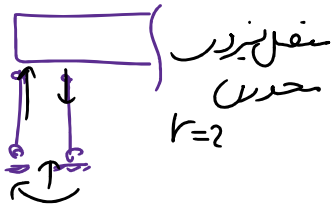
⑤ گیردار

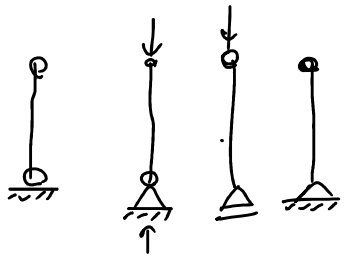


⑥ تیرها



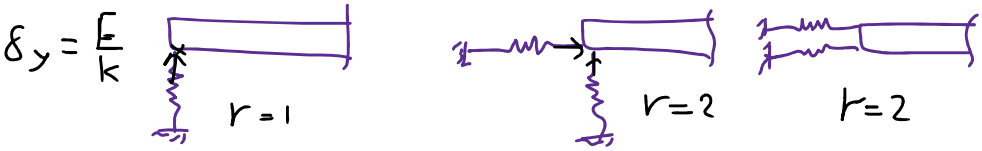




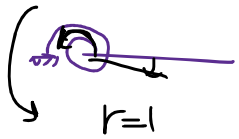


⑦ تکیه گاه ارتجاعی

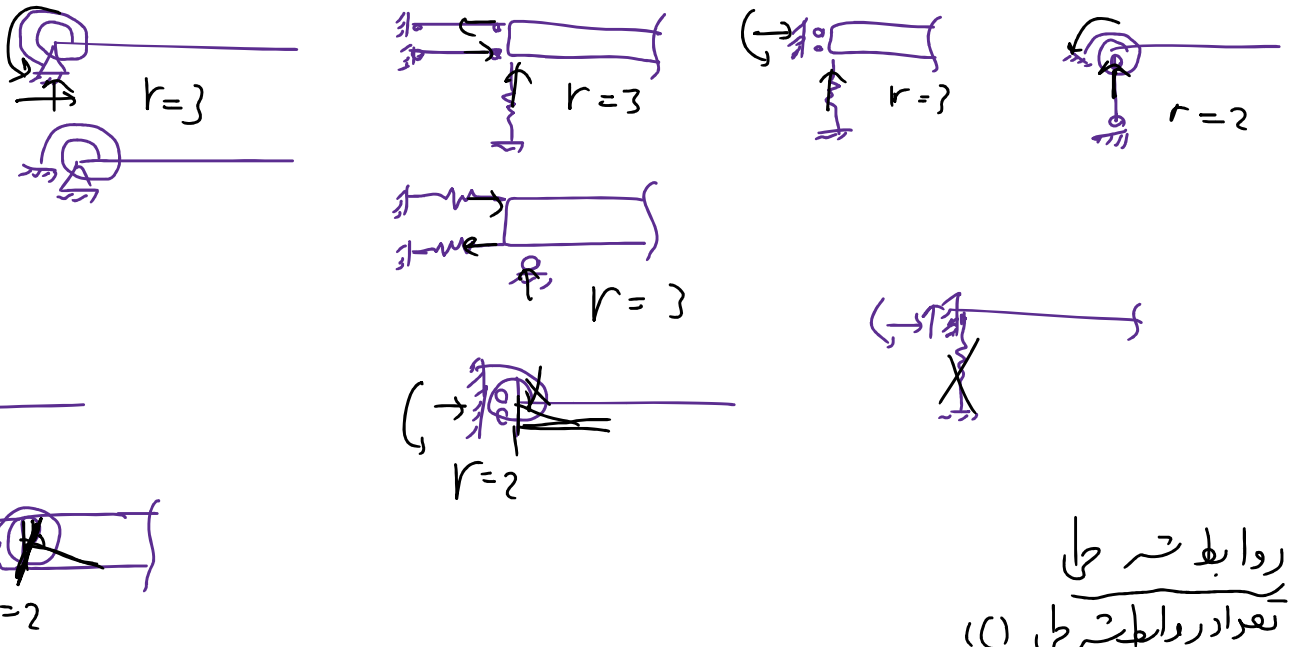
۱- فنر محدود



۲- فنر چرخشی



⑧ تکیه گاه ترکیبی

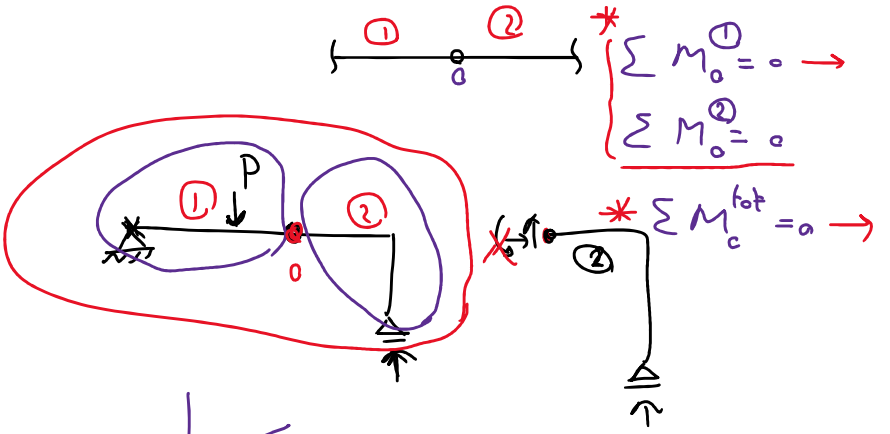


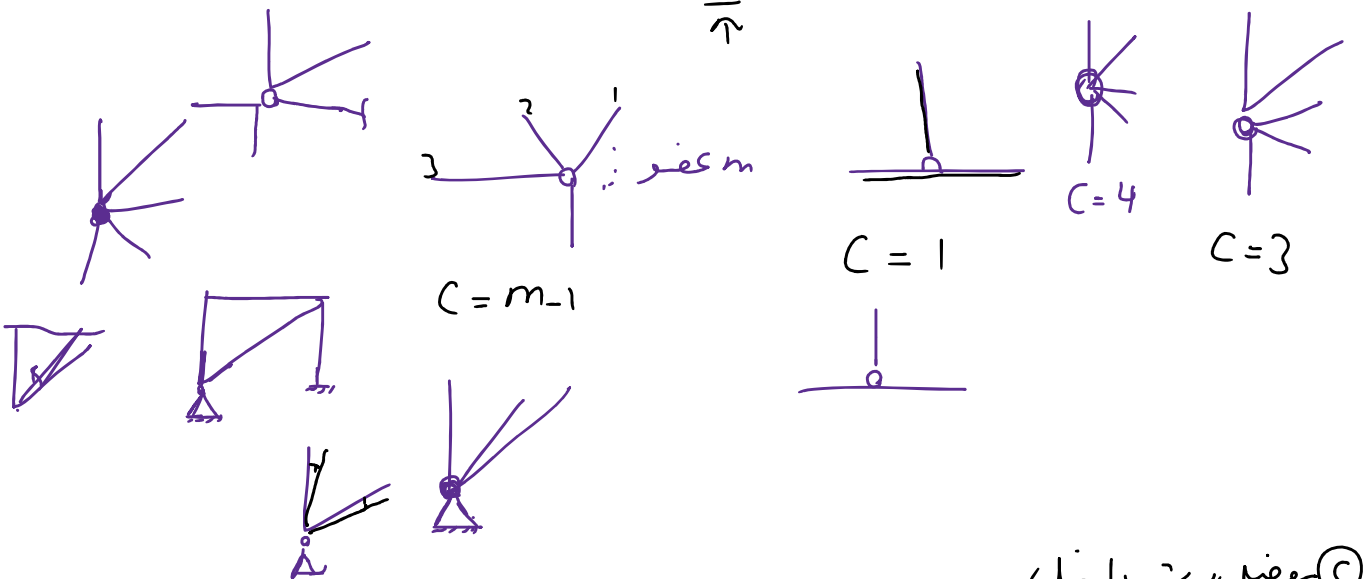
روابط شرطی

تعداد روابط شرطی ( )

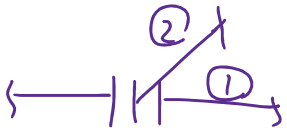
① مفصل داخلی

$C = 1$

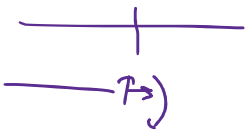




③ مفصل برشی داخلی

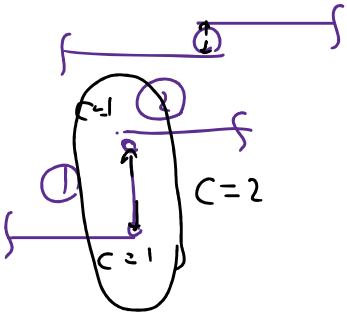


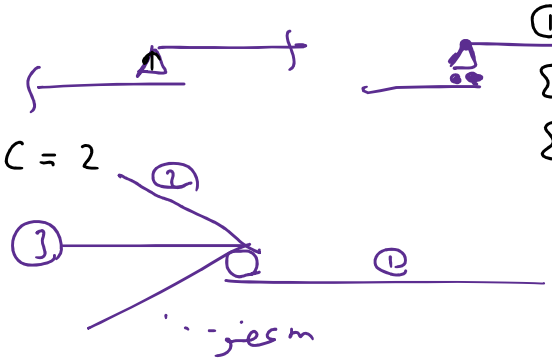
② مفصل محوری داخلی

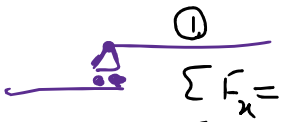


$\sum F_x = 0 \quad C = 1$

① تکیک داخلی





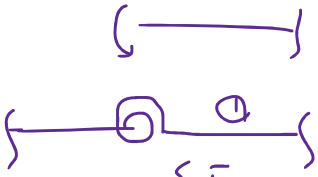


$\sum F_x = 0$

$\sum M = 0$

$C = 2(m-1)$

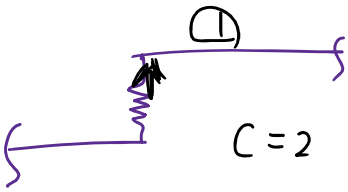
③ فنر داخلی



$C = 2$

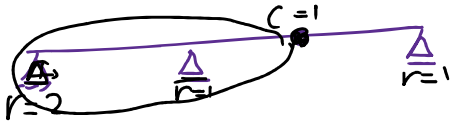
$\sum F_x = 0$

$\sum F_y = 0$

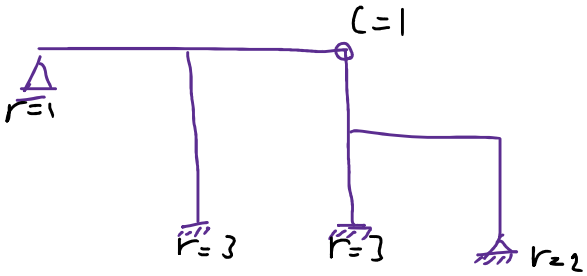


$C = 2$

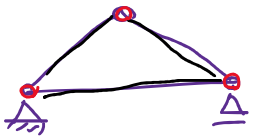
مثال:



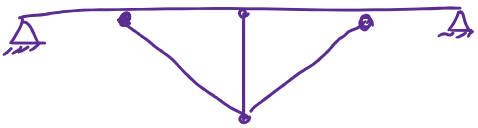
$n = 4 - (3+1) = 0$

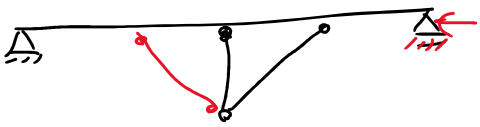


$n = 9 - (3+1) = 5$

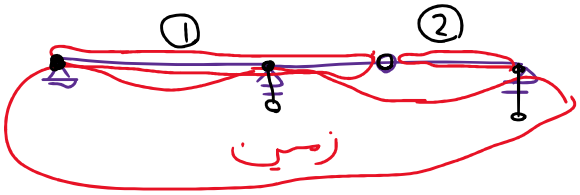


$n = 0$

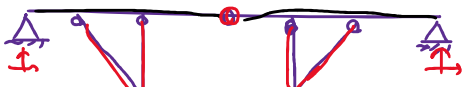




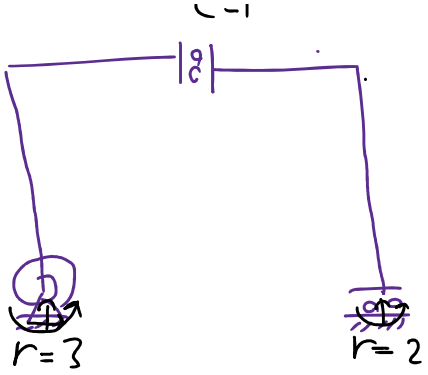
$n = 2$



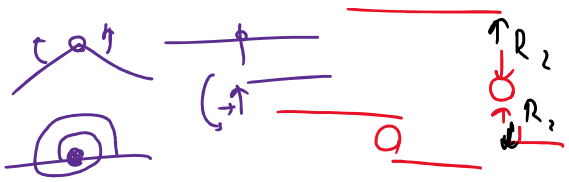
$C = 1$

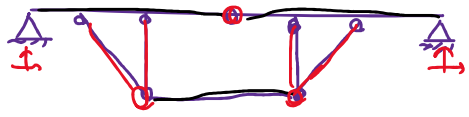


$n = 1$

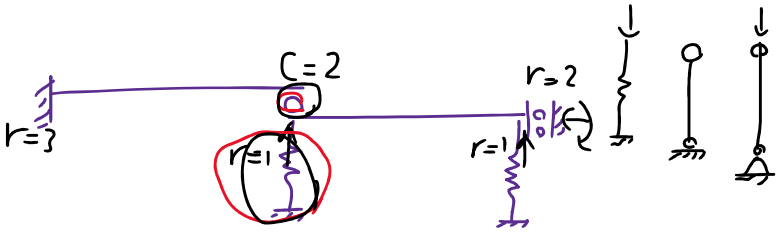


$n = 5 - (3+1) = 1$

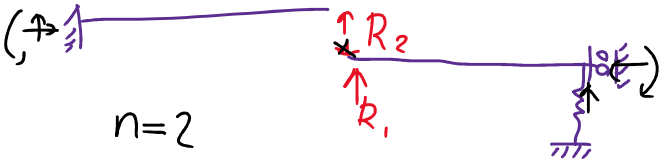




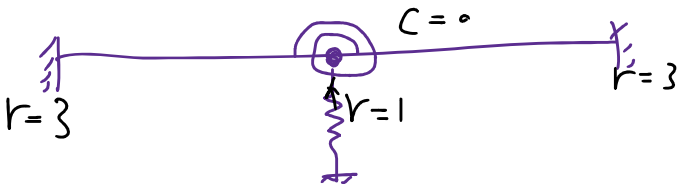
$n = 1$



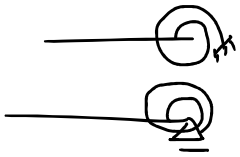
$n = 7 - (3+2) = 2$



$n = 2$

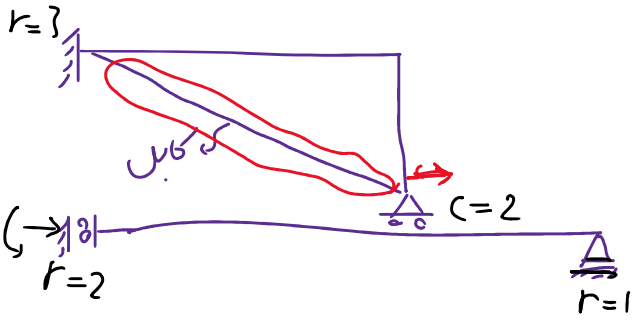


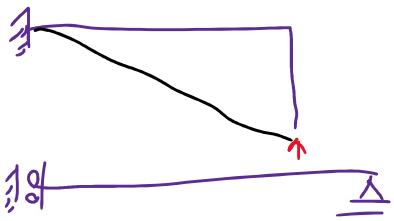
$n = 7 - (3+0) = 4$



کابل درشت $n = 2$

کابل درنشار $n = 6 - (3+2) = 1$





$n = 1$

$n = 2$

Determ

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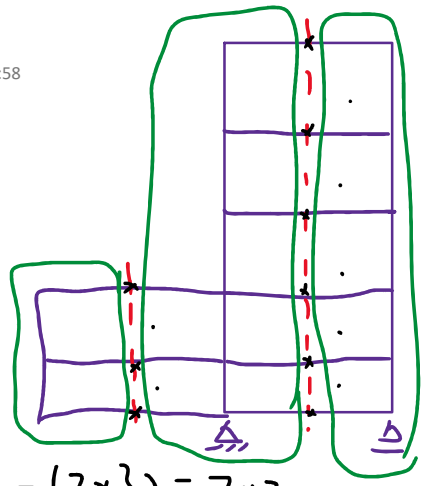
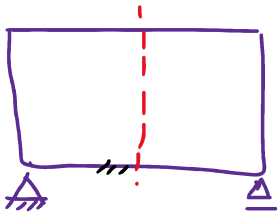
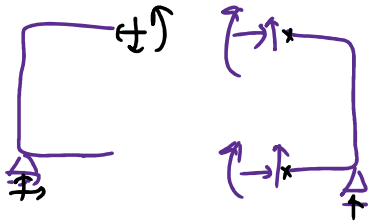
حلقه بسته



$h = (3 + 9\times3) - (3\times3) = 7\times3$



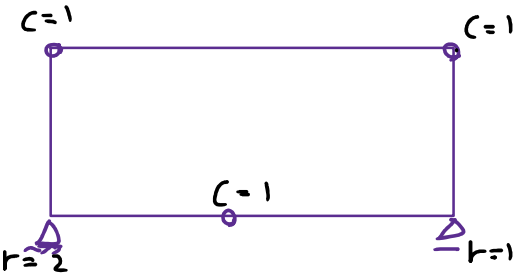


$h = (3 + 6\times3) - (2\times3) = 5\times3$

$n = (3 + 2\times3) - (2\times3) = 3$

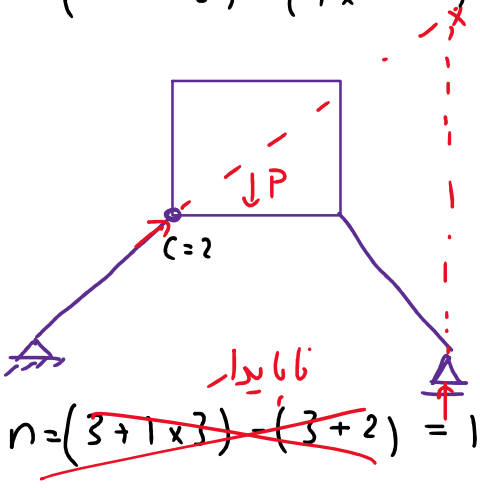
$$n = (3,0 + r) - (3 + c)$$

تعداد حلقه بسته — تعداد واکنش‌ها — تعداد میله‌ها — معادلات شرط

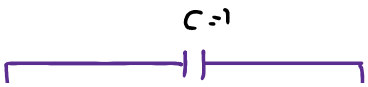


$h = (3 + 1\times3) - (3 + 3) = 0$

$n = (3\times4 + 3) - (4\times3 + 3) = 0$

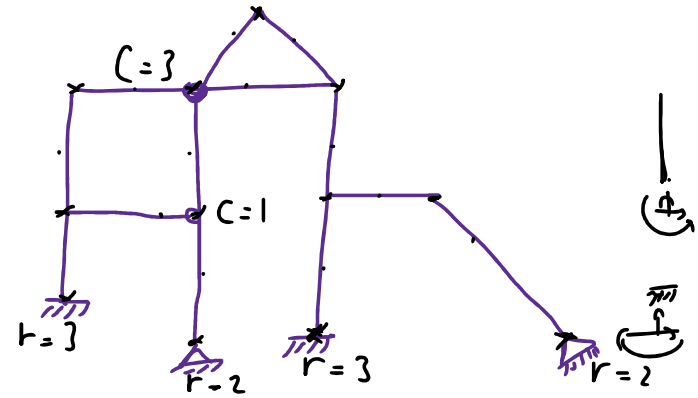


$n = (3 + 1\times3) - (3 + 2) = 1$



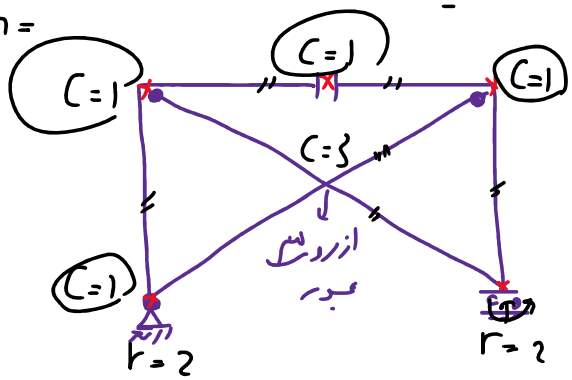
مثال:

$n = (13\times3 + 10) - (12\times3 + 4) = 9$



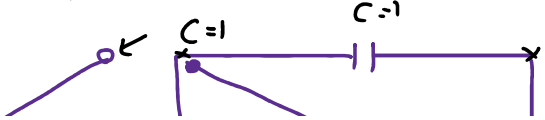
$n = (10 + 2\times3) - (3 + 4) = 9$

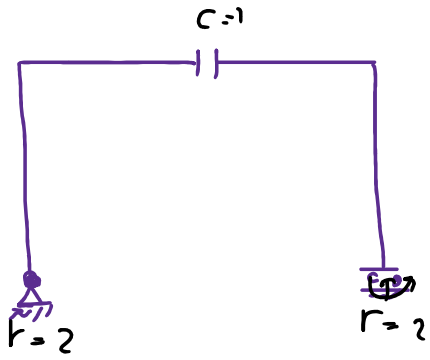
$n =$



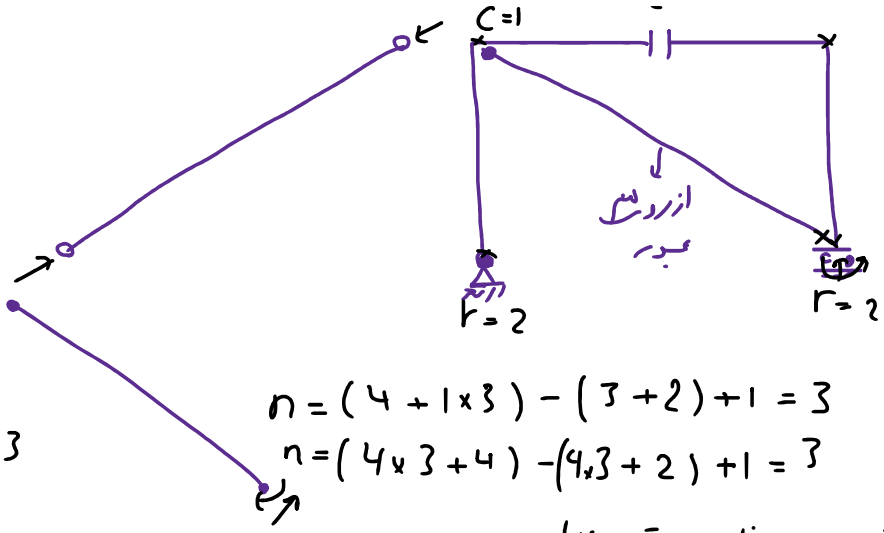
$n = (4 + 3\times3) - (3 + 7) = 3$

$n = (6\times3 + 4) - (5\times3 + 4) = 3$





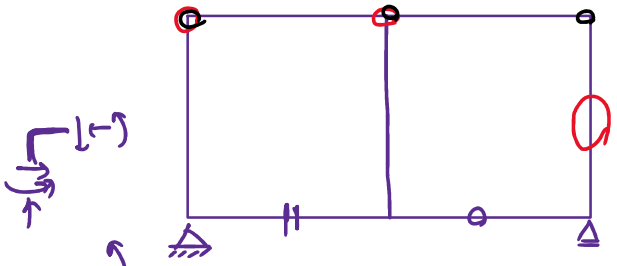
$n=(4+0)-(3+1)+1+2=3$



$n=(4+1\times3)-(3+2)+1=3$

$n=(4\times3+4)-(4\times3+2)+1=3$

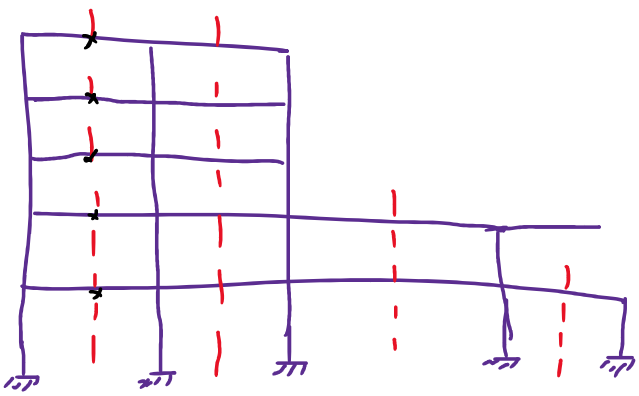
رابطه دوم درجه نامعین تیرها

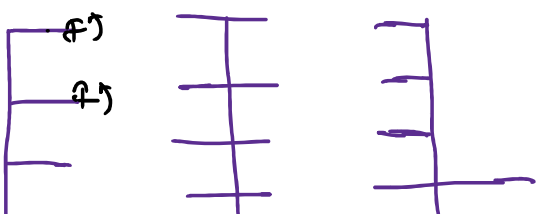


$n=(3m+r)-(3j+c)$

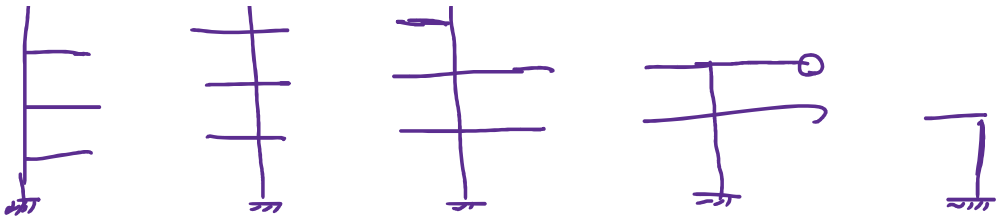
m: تعداد اعضا
r: تعداد واکنش‌ها
j: تعداد گره‌ها
c: ...

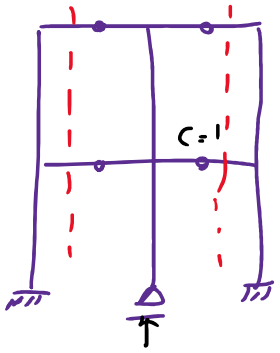
روش درختی برای تیرها



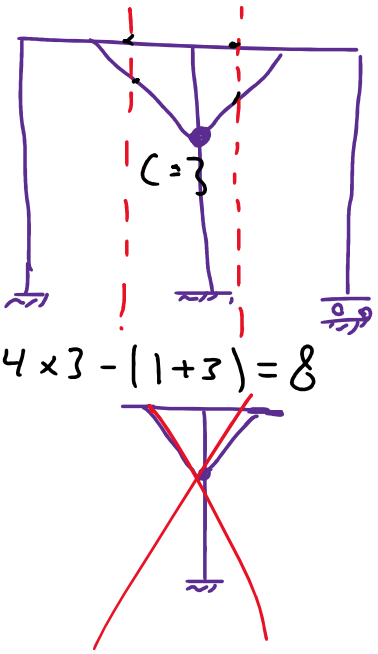


$n=13\times3=39$





$n = (4 \times 3) - (2 + 4) = 6$



$n = 4 \times 3 - (1 + 3) = 8$

Truss types

Saturday, October 14, 2023 14:19

پایداری و معین خرپاها

انواع خرپاها :

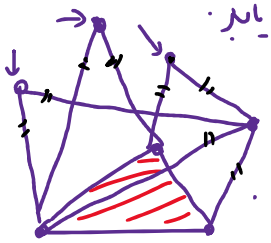
۱) خرپای ساده

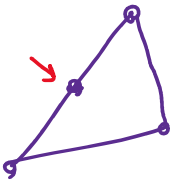
۲) خرپای مرکب

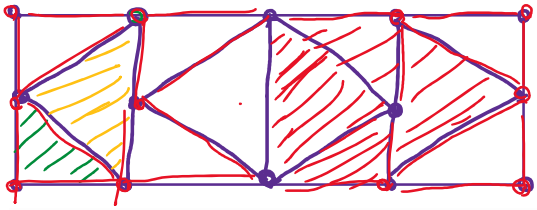
۳) خرپای پیچیده (لغزنی)

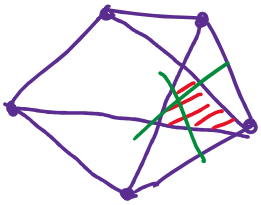
۱) خرپای ساده

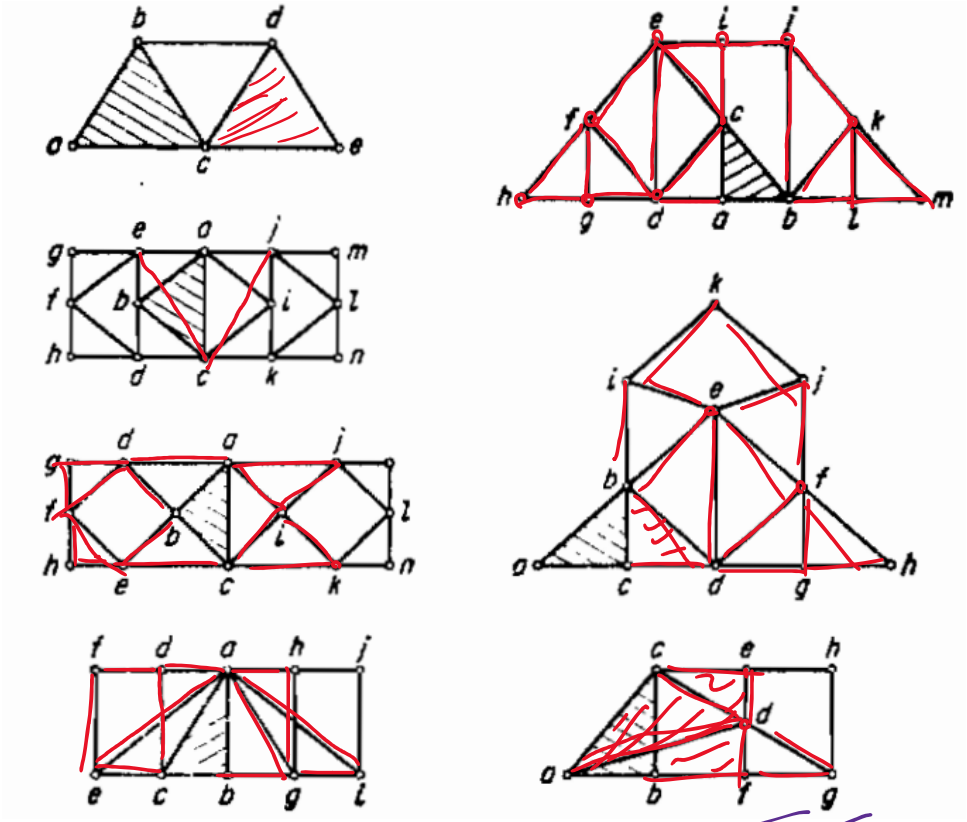
از ترکیب پایدار یک میله و یک مفصل (یک مثلث) شروع می شود و با دو میله و یک مفصل گسترش می یابد.





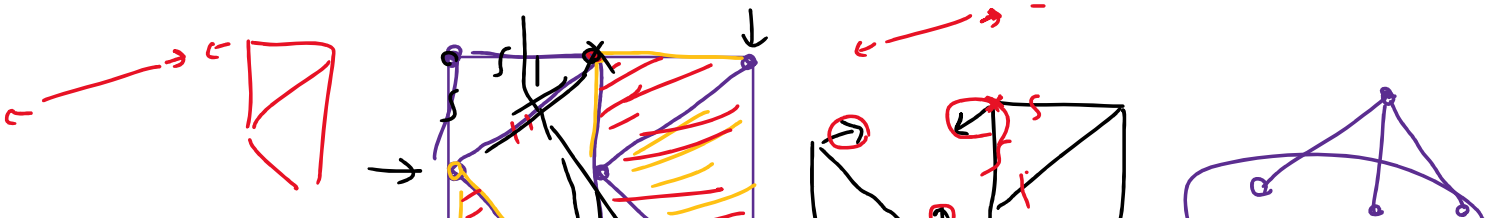


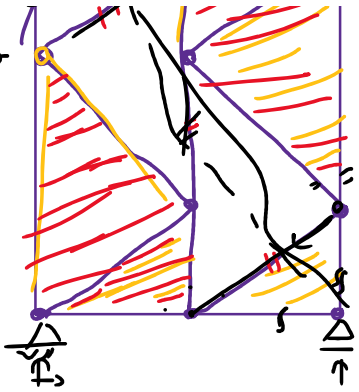


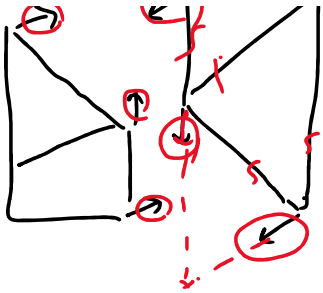


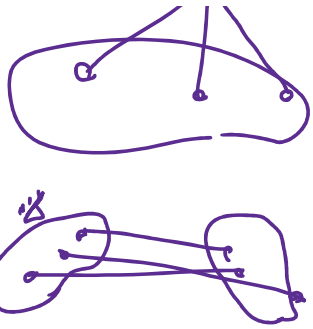
۲) خرپای مرکب

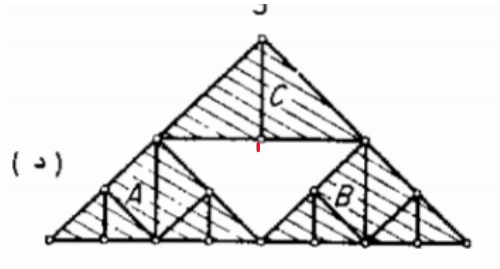
خرپایی است که از ترکیب پایدار دو یا چند خرپای ساده ایی حاصل می شود.

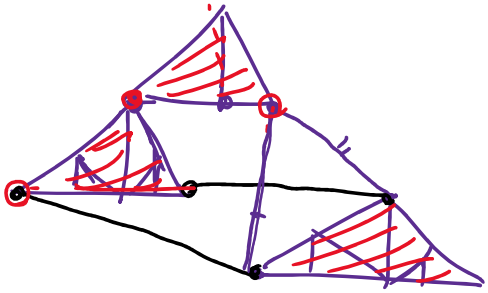






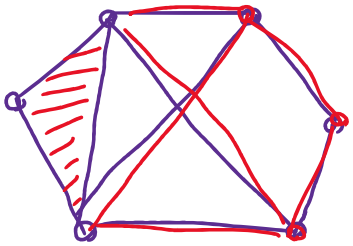




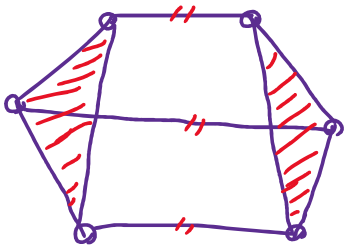


② خرپا پیچیده (غیرساده)

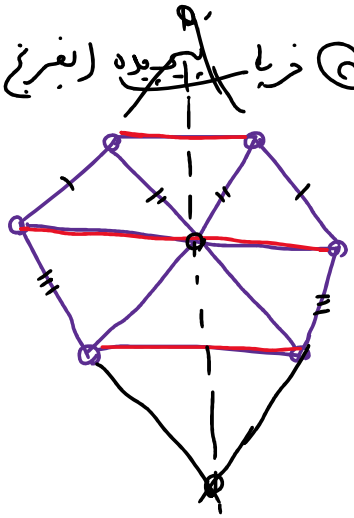
خرپایی که دارای مشخصات خرپای ساده و مرکب نباشد.

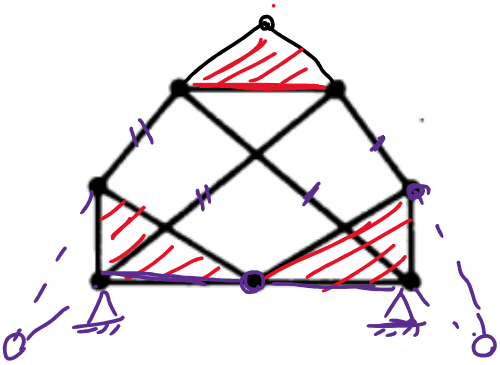


ساده



مرکب





$$n = (m + r) - 2j$$

n: تعداد مجهولات

m: تعداد میله‌ها

r: تعداد عکس‌العمل‌ها

j: تعداد مفاصل

تعداد مناطق

تعداد محصول
تکرارها

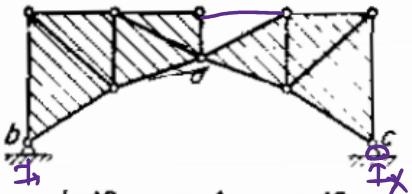
تعداد ایتمها

Truss Ex

Tuesday, October 17, 2023 20:10

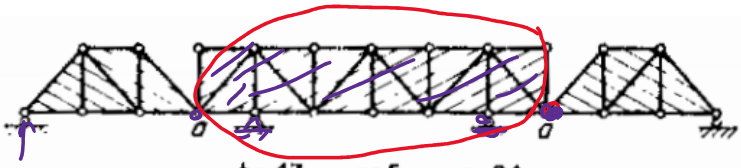
مثال: درجه نامعینی و پایداری خرپاهای زیر را بررسی کنید.

$16+4-10(2)=0$



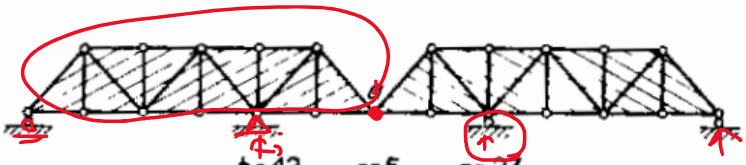
$b=16 \quad r=4 \quad n=10$

(الف)



$b=43 \quad r=5 \quad n=24$

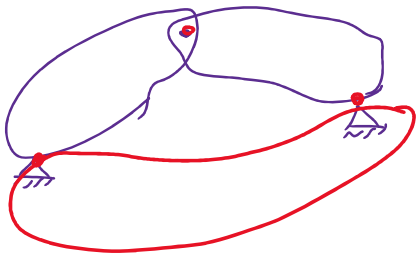
(ب)

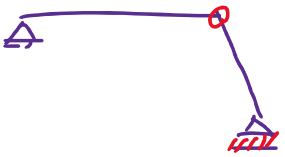


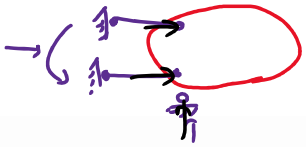
$b=42 \quad r=5 \quad n=23 \quad n=1$

(ج)

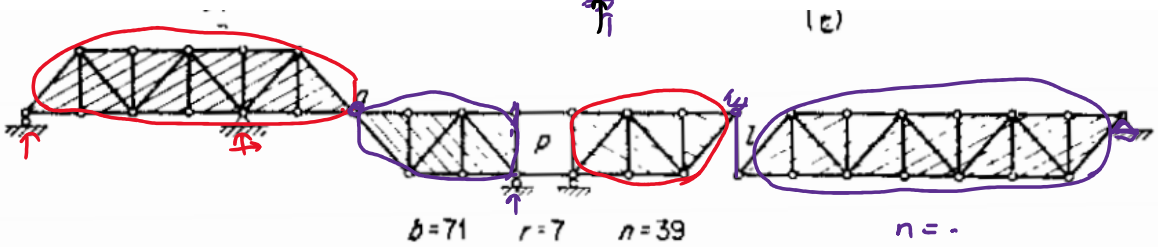
$42+5-46=1$



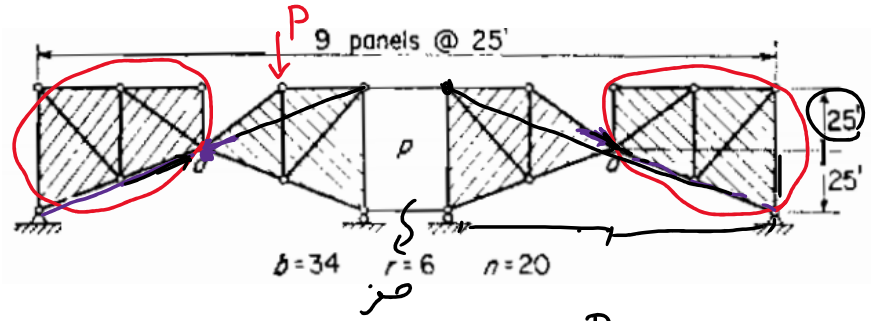




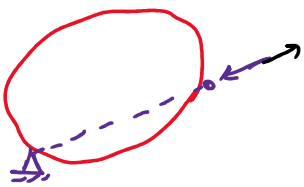
(د)

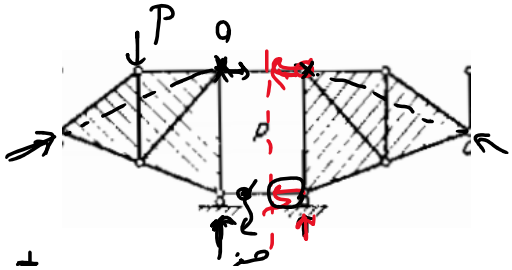


$b=71 \quad r=7 \quad n=39 \quad n=0$

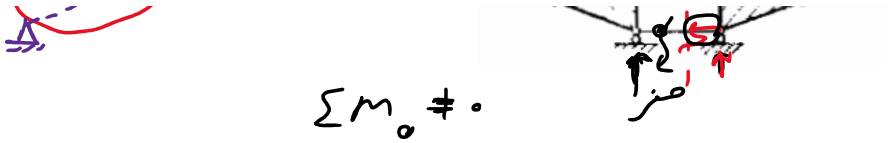


صفر

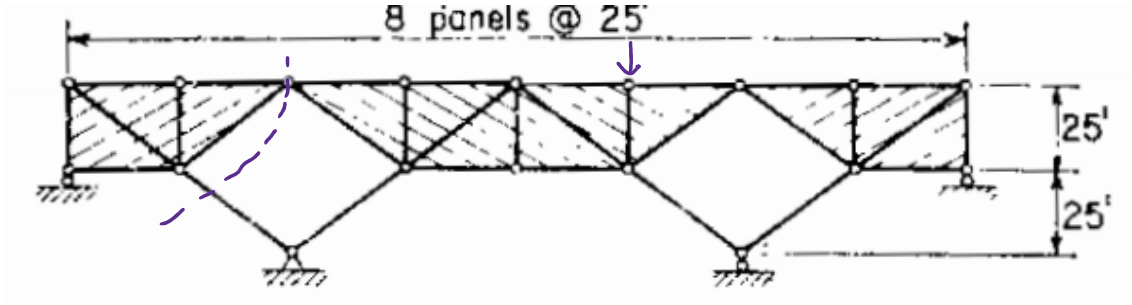


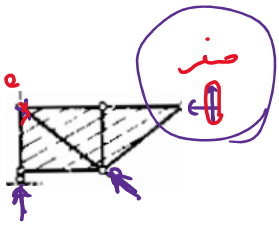


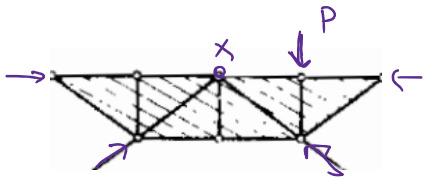
صفر

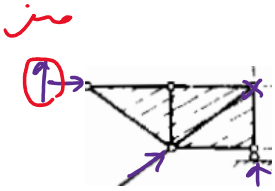


$\Sigma M_o \neq 0$









$\Sigma M_x \neq 0 \rightarrow$

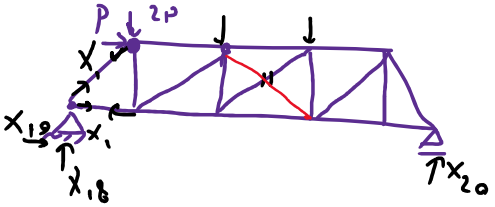
$\Sigma M_o = 0 \rightarrow P(2a) = 0 \rightarrow P = 0$

Zero load

Thursday, October 19, 2023 12:28

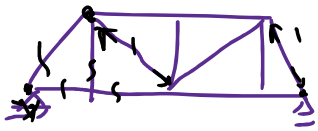
بررسی ناپایداری خرپاها به روش درجه‌ی نامعینی

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1,20}x_{20} = P_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2,20}x_{20} = P_2 = 0 \\ \vdots \\ a_{20,1}x_1 + \cdots = P_{20} = 0 \end{cases}$$



$$[A]\{X\} = \{P\} \rightarrow \begin{cases} |A| \neq 0 \rightarrow \text{جواب یکتا} \rightarrow \text{پایدار} \\ |A| = 0 \rightarrow \text{جواب یکتا ندارد} \rightarrow \text{ناپایدار} \end{cases}$$

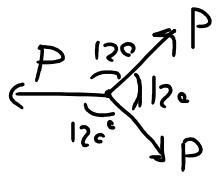
بررسی ناپایداری خرپاها به روش بار صفر

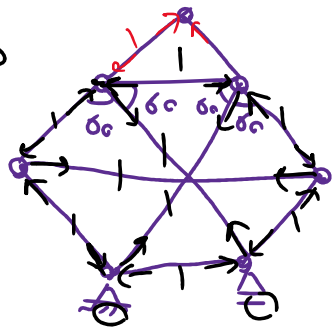


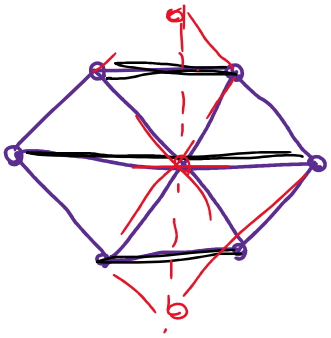
در یک سازه معین، اگر بارگذاری خارجی نداشته باشیم، نیروی همه‌ی اعضا باید صفر باشد.

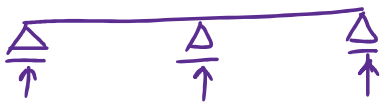
حال اگر بتوان یک دسته جواب غیرصفر پیدا کرد (بدون آنکه هیچ تناقضی در معادلات تعادل ایجاد شود) می‌توان نتیجه گرفت که سازه یکپارچه ناپایدار است.

* روش بار صفر نقاط ناپایداری سازه را اثبات می‌کند.

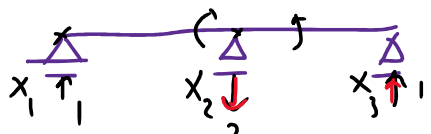




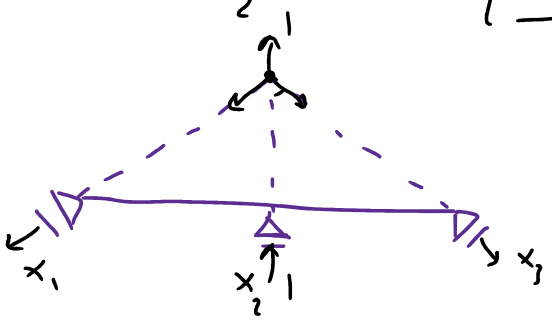


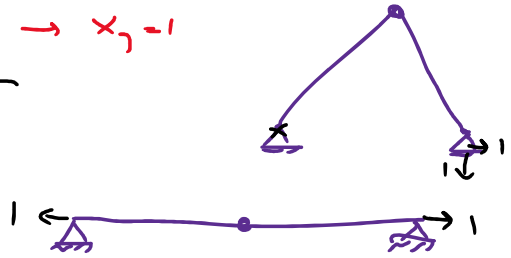


$n = 0$

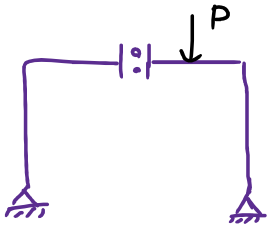


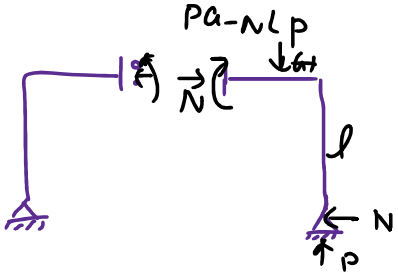
$$\begin{cases} X_1 + X_2 + X_3 = 0 & X_2 = -2 \\ X_1 L - X_3 L = 0 \rightarrow & X_3 = 1 \end{cases}$$

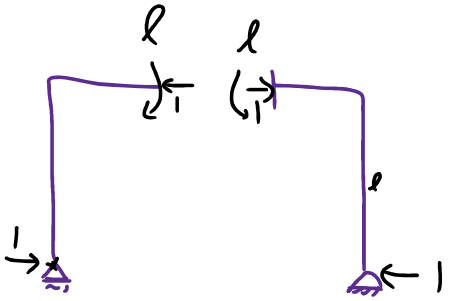


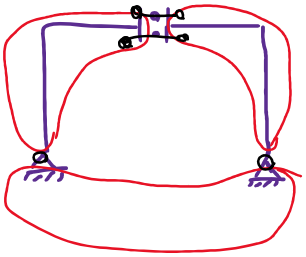


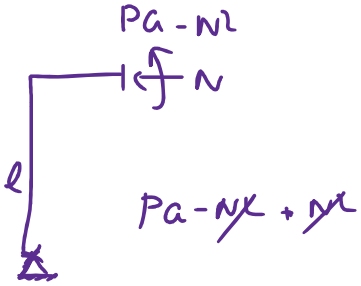
ناپایدار

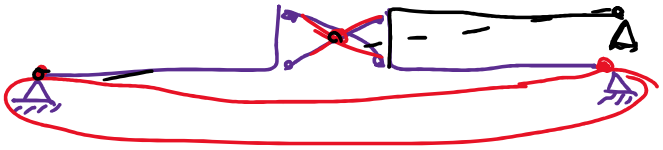


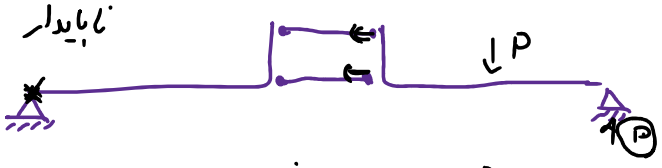


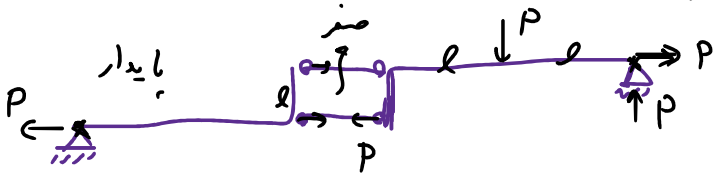


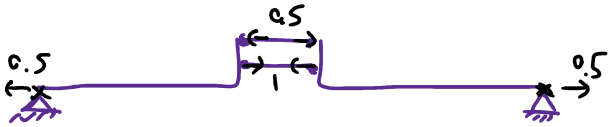


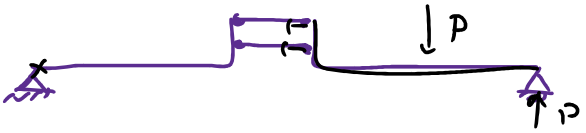






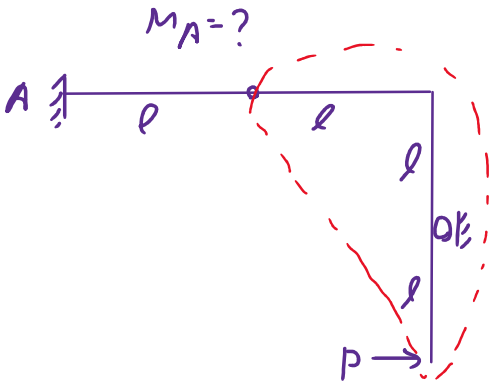


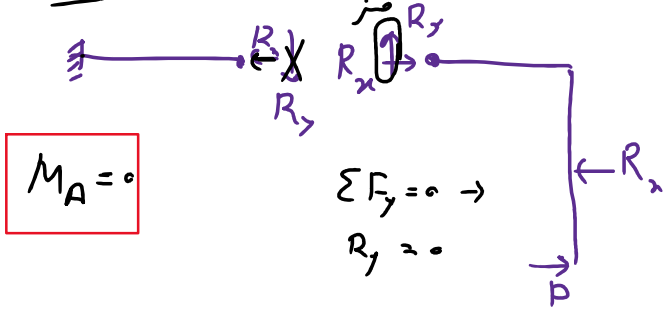


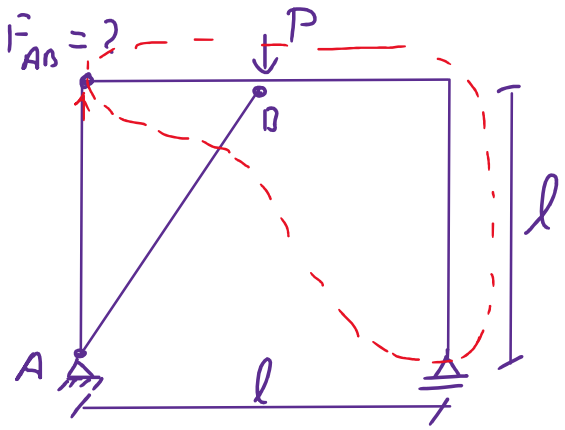


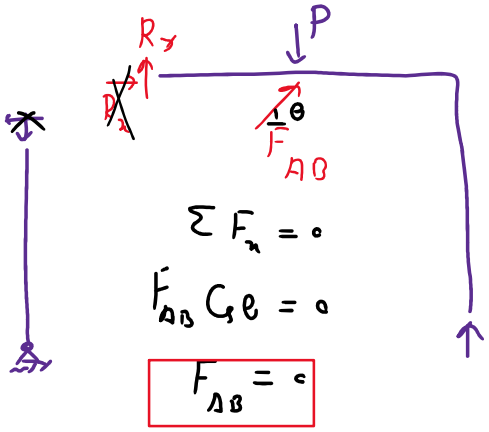
فصل دوم: تحلیل سازه های معین

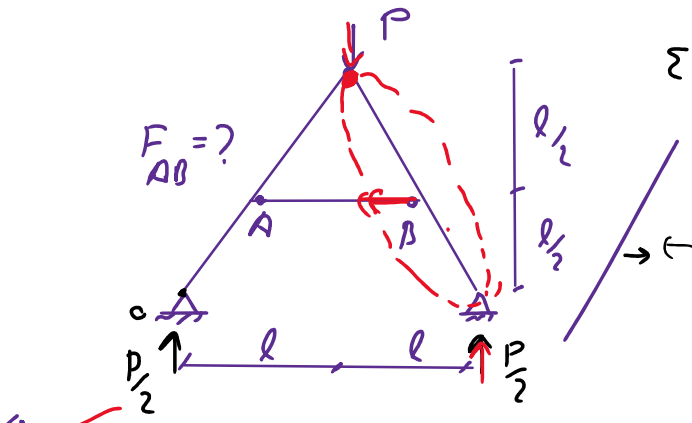
استفاده از معادلات تعادل و دیاگرام آزاد



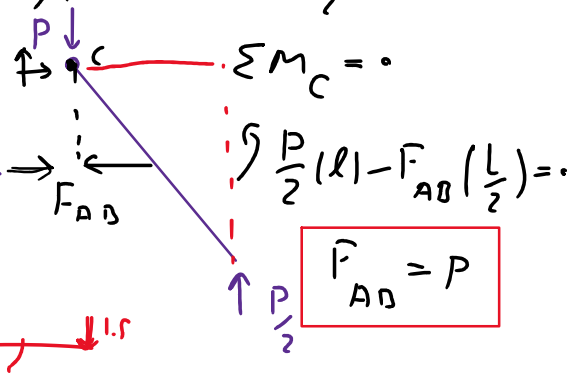


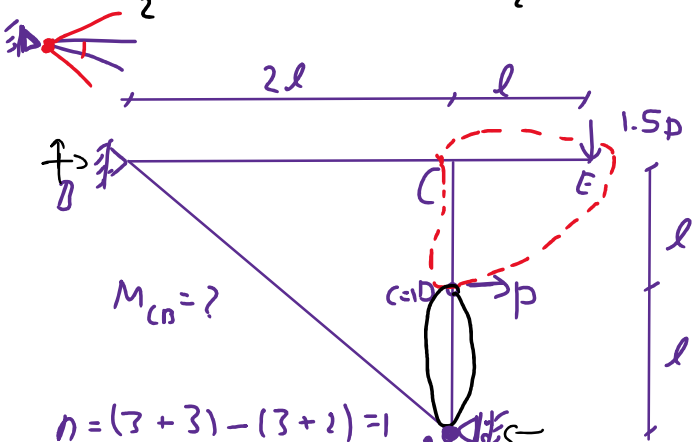


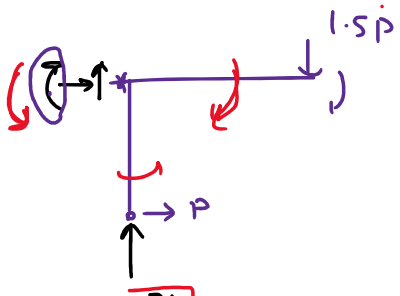




$\Sigma M_? = 0 \rightarrow PL - R_y(2l) = 0 \rightarrow R_y = \frac{P}{2}$

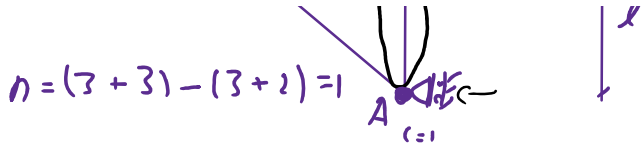






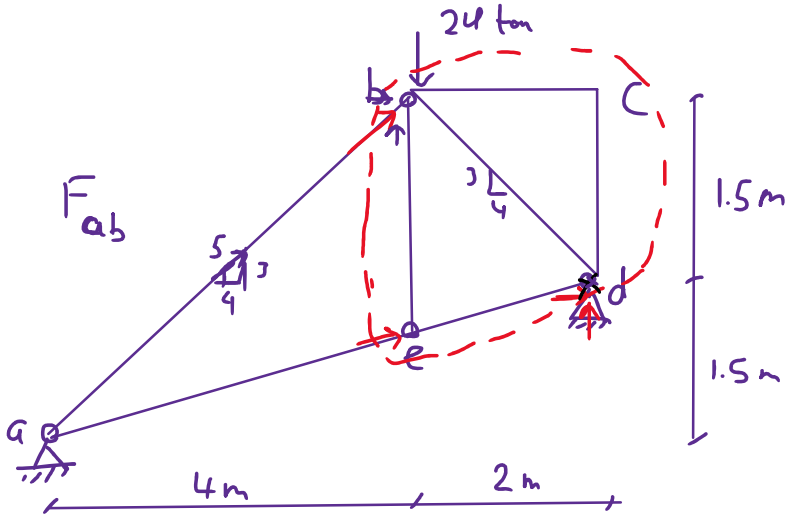
$M_{CD} = 1.5P(l) - Pl = \frac{PL}{2}$

$n = (3+3) - (3+2) = 1$



$M_{CB} = 1.5P(\ell) - P\ell = \frac{PL}{2}$

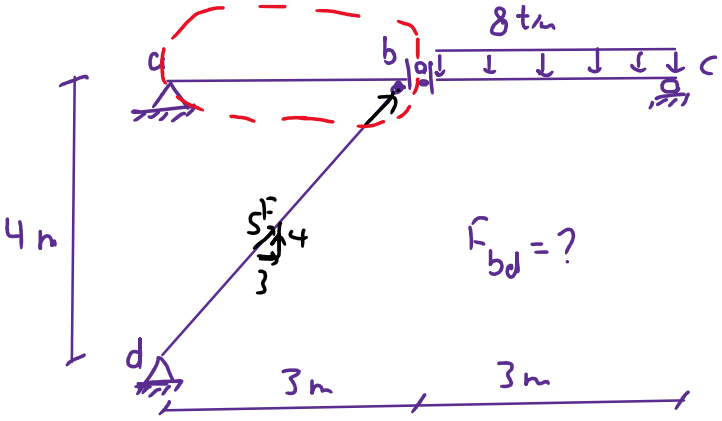
$1.5P(\ell) - P\ell + M_{CB} = 0 \rightarrow M_{CB} = -\frac{PL}{2}$

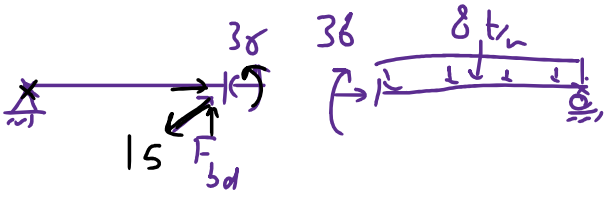


$\Sigma M_d = 0$

$24(2) - F\left(\frac{4}{5}\right)(1.5) - F\left(\frac{3}{5}\right)(2) = 0$

$F_{ab} = 20$ ton

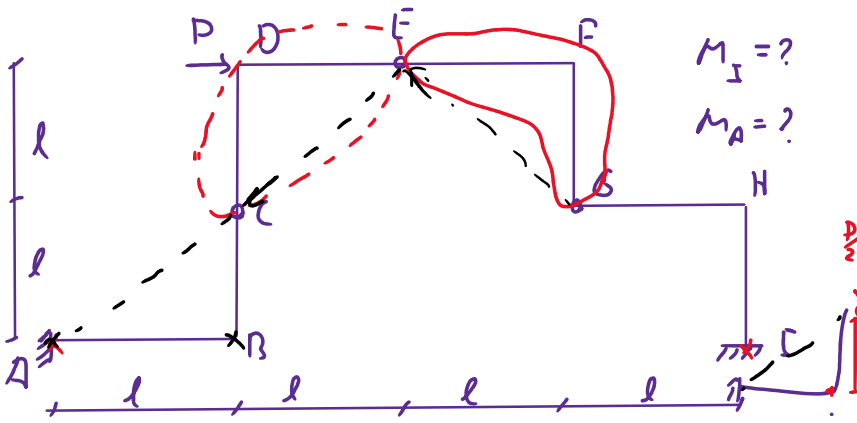




$8(3)(1.5) - M = 0$

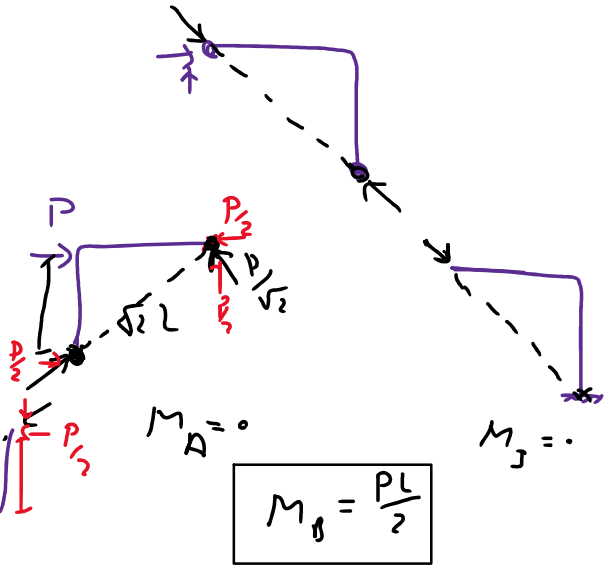
$\frac{4}{5}F(3) + 36 = 0 \rightarrow F_{bd} = -15$

P, V, M

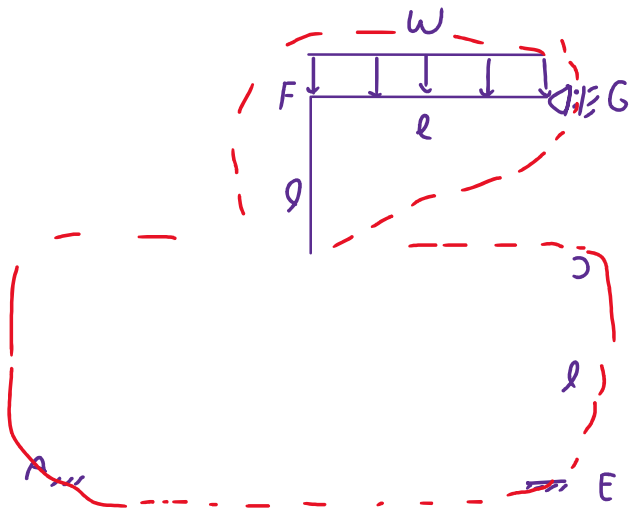


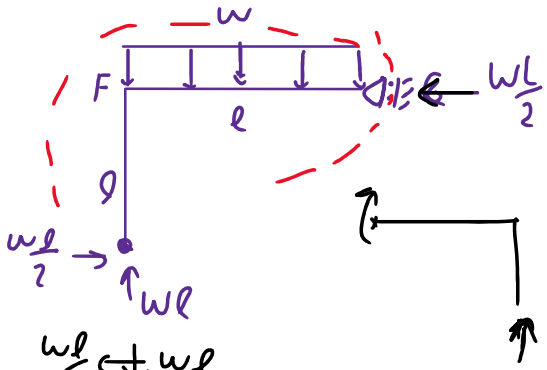
$M_I = ?$

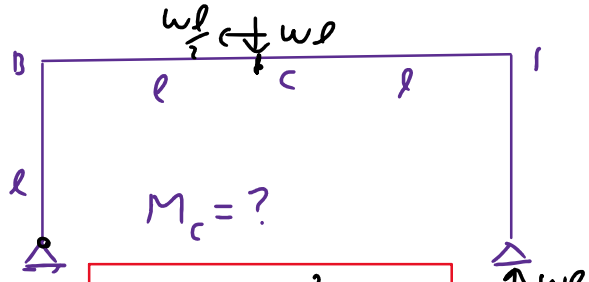
$M_A = ?$



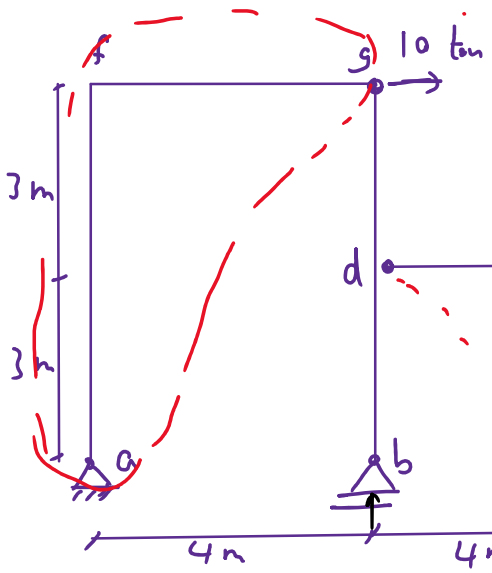
$M_B = \frac{PL}{2}$

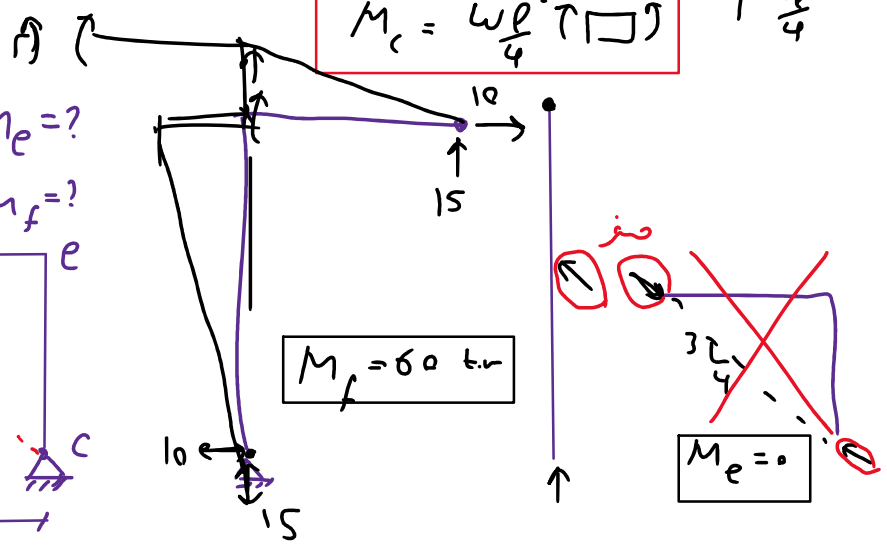


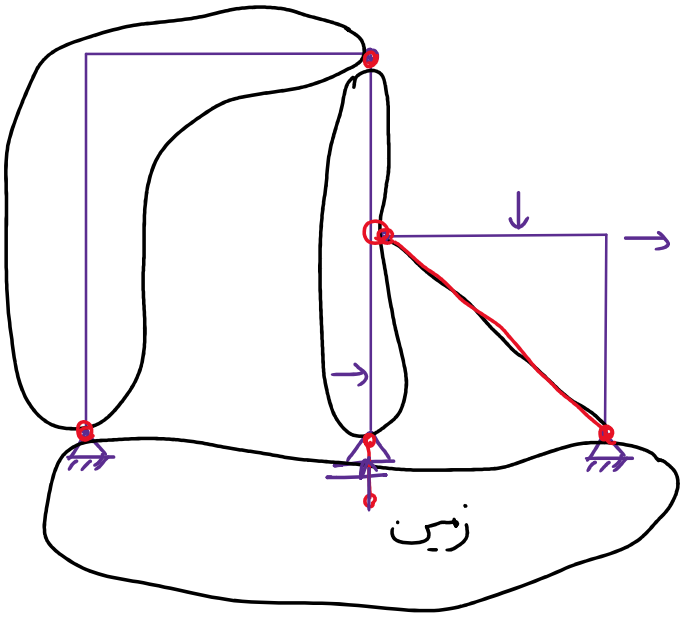


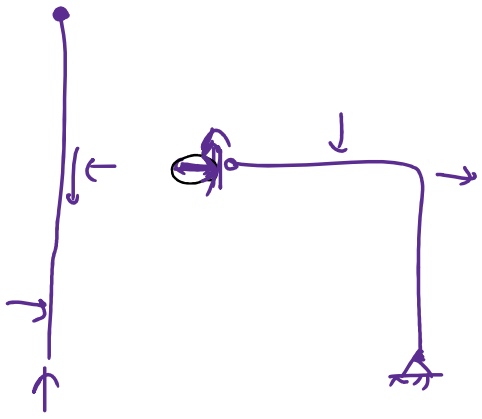


$$M_c = \frac{w\ell^2}{4}$$









Beams

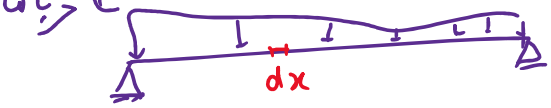
Sunday, October 22, 2023 17:15

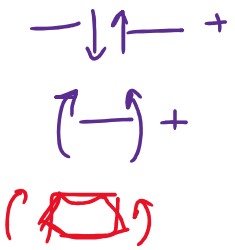
تحلیل سازه‌های معین

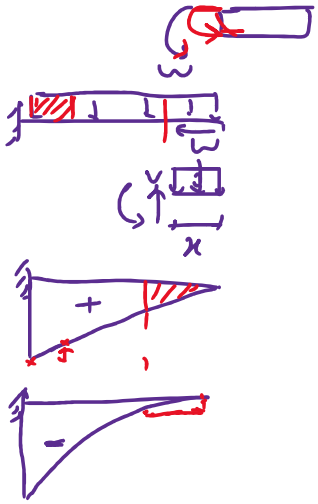
۱) تیر

۲) قاب‌ها

۳) خرپاها

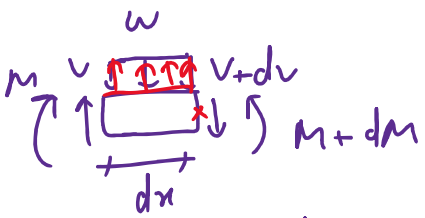






$V = wx$

$M = \frac{wx^2}{2}$



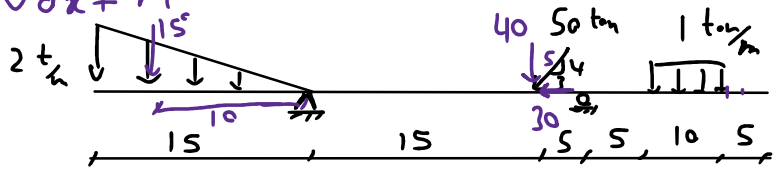
$$\Sigma F_y = 0 \rightarrow V + w\,dx - (V + dV) = 0 \rightarrow \frac{dV}{dx} = w$$

$$\Sigma M = 0 \rightarrow V\,dx + M + w\frac{dx^2}{2} - (M + dM) = 0 \rightarrow$$

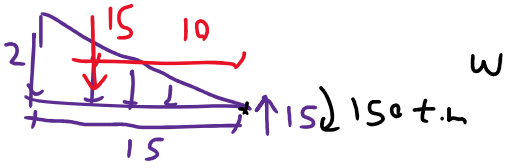
$$\frac{dM}{dx} = V$$

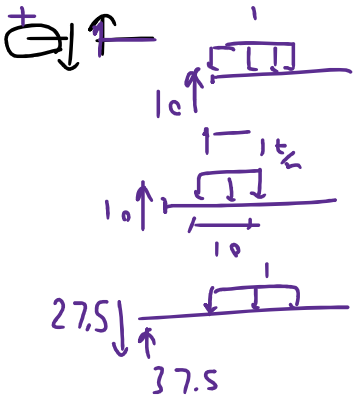
$$\begin{cases} \frac{dV}{dx} = w \\ \frac{dM}{dx} = V \end{cases} \rightarrow \begin{cases} V = \int w\,dx + V_0 \\ M = \int V\,dx + M \end{cases}$$

مثال:

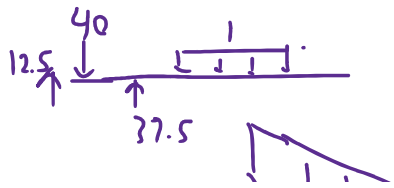


$40 \times 15 + 10 \times 30 - 15 \times 10 = 20R$

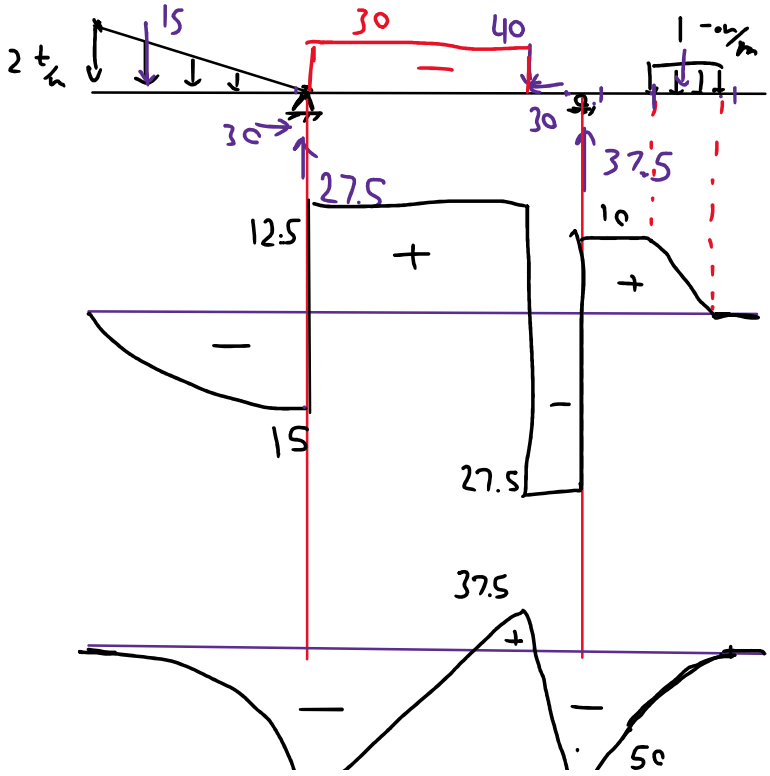


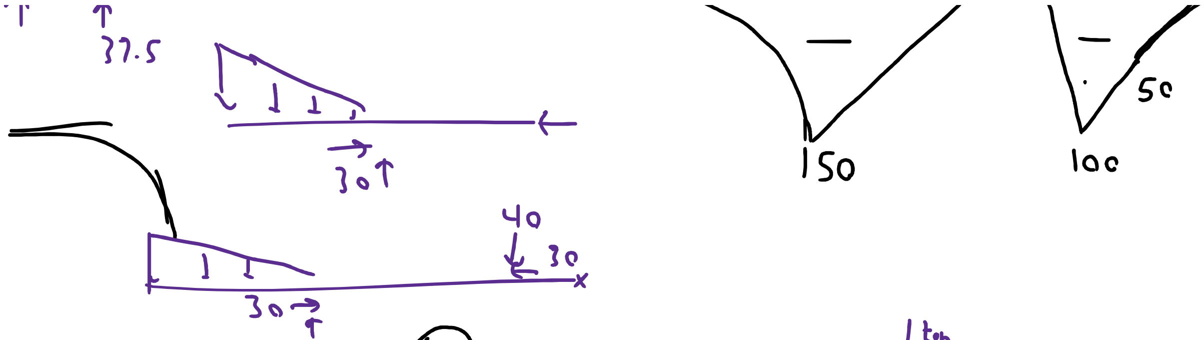


(V)

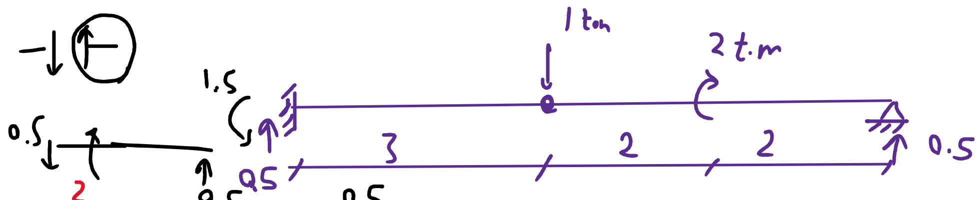


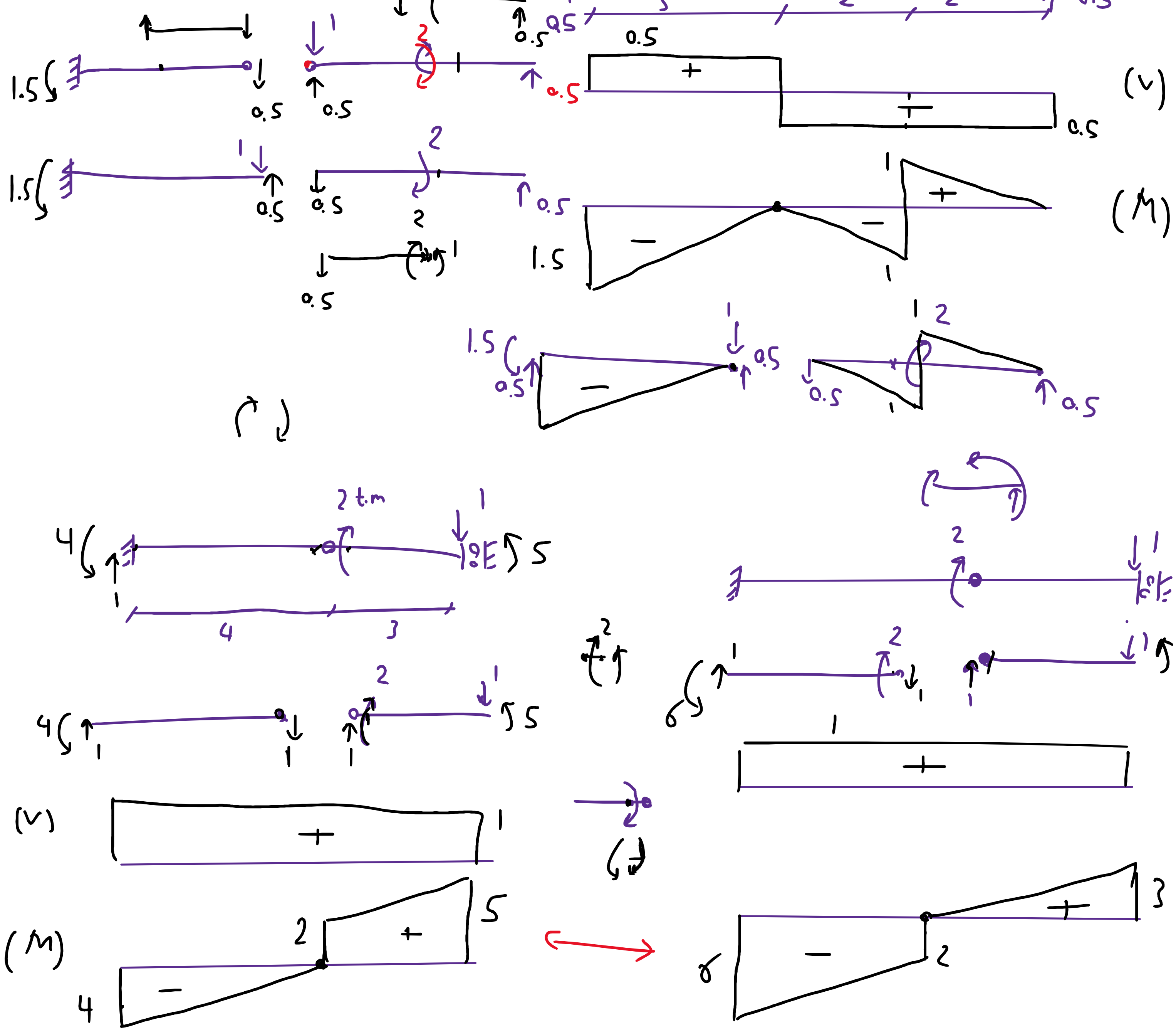
(M)





$n = 3 - (2 + 1) = 0$

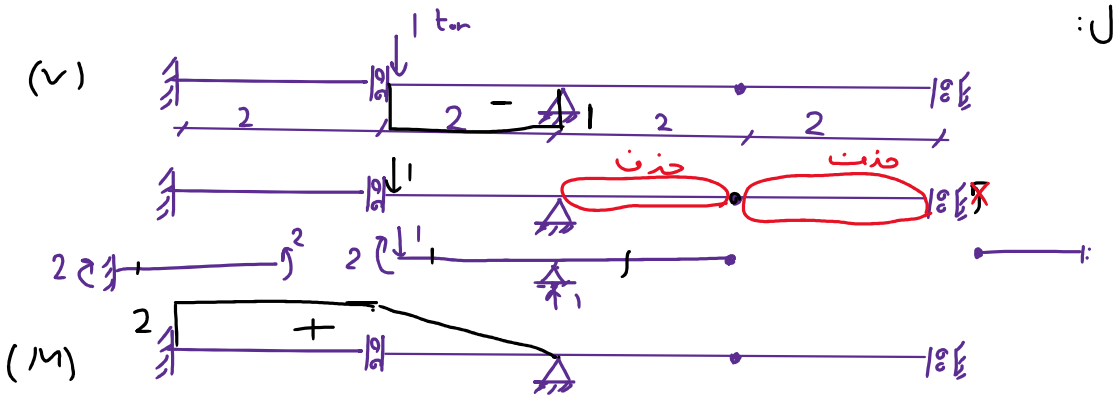




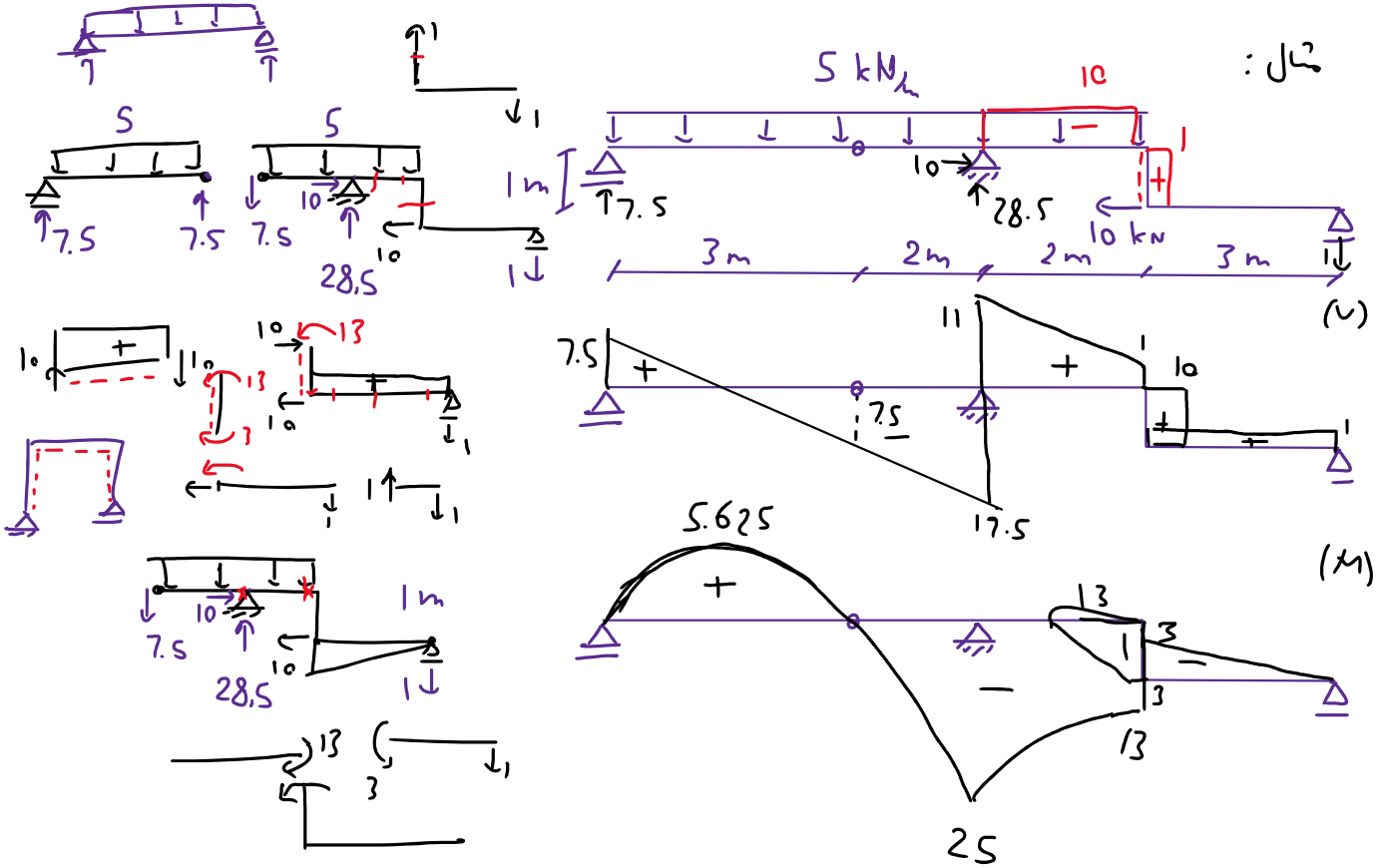
Beams2

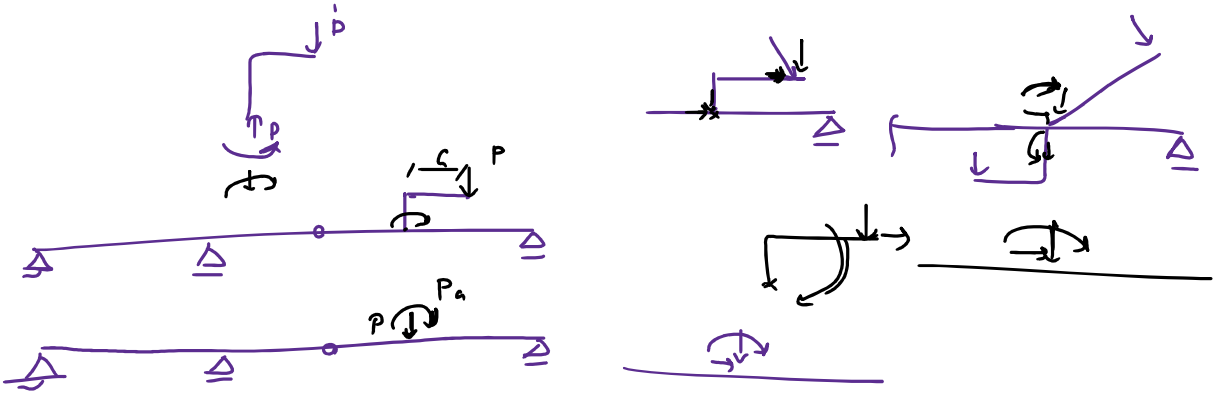
Monday, October 23, 2023 17:05

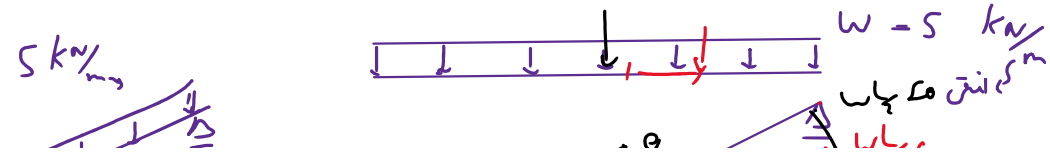
مثال:

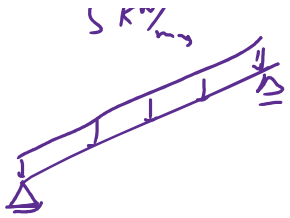


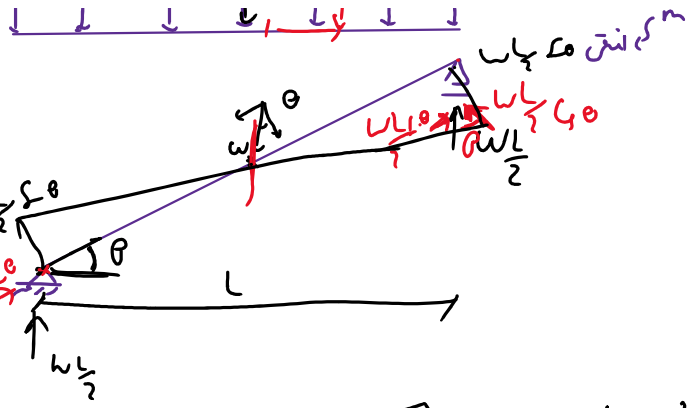
مثال:



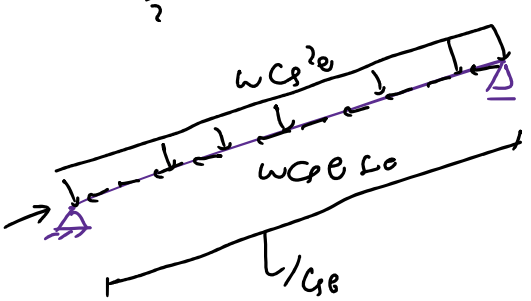








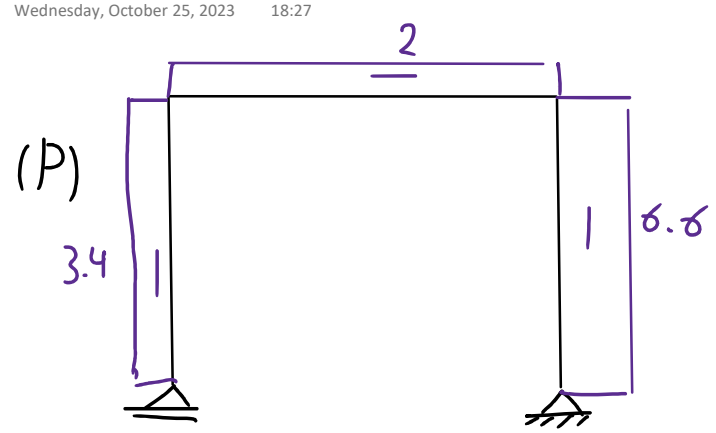
$$\frac{wk}{\frac{k}{\cos\theta}} = w\cos\theta$$

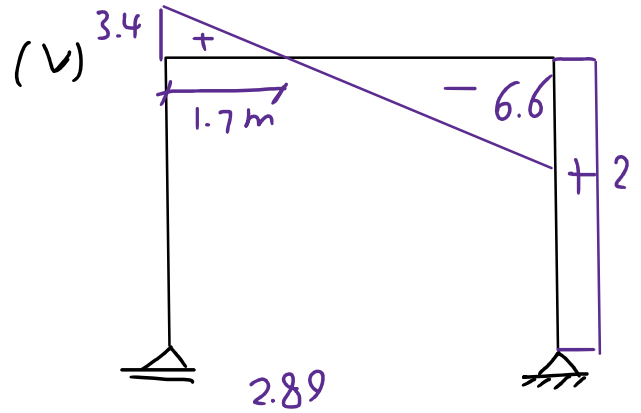


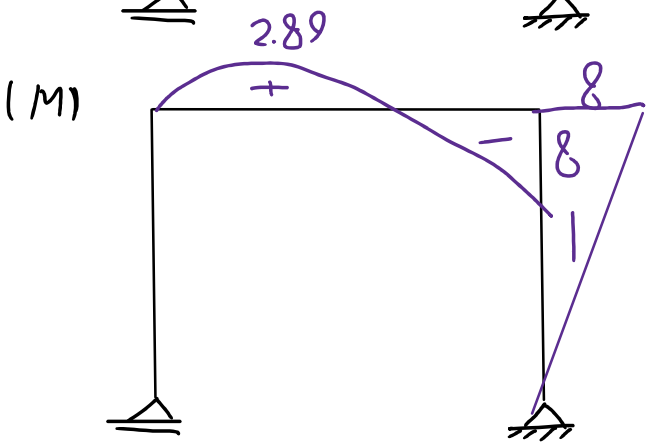
$$\frac{w\cos^2\theta\left(\frac{L}{\cos\theta}\right)^2}{8}$$

Frame

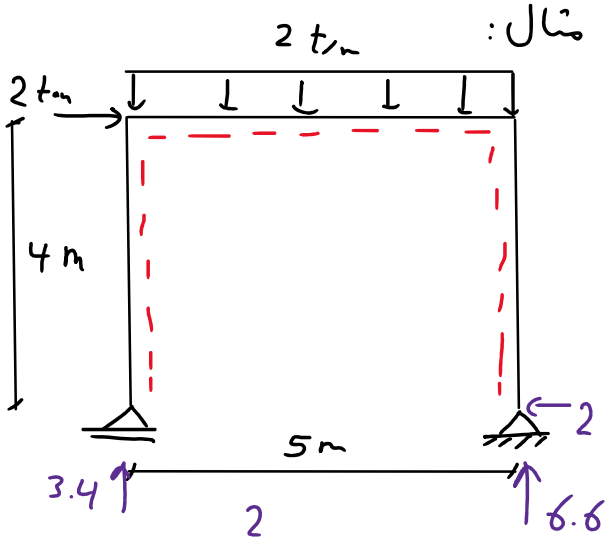
Wednesday, October 25, 2023 18:27

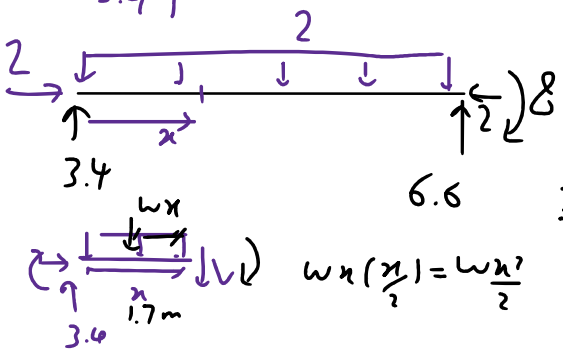






مثال :

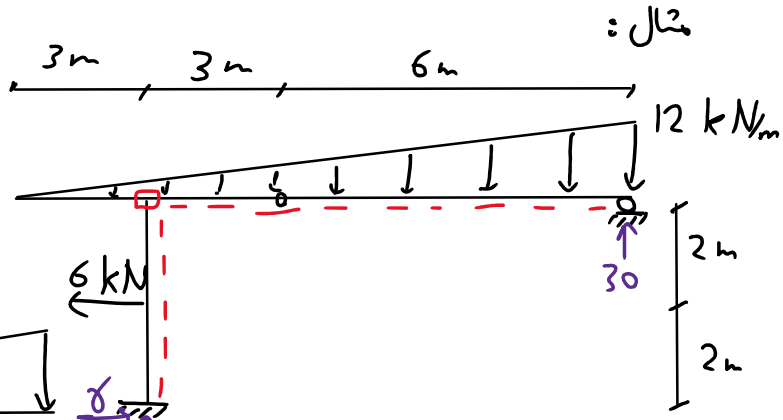


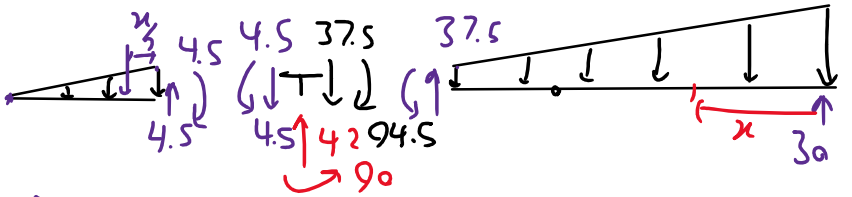


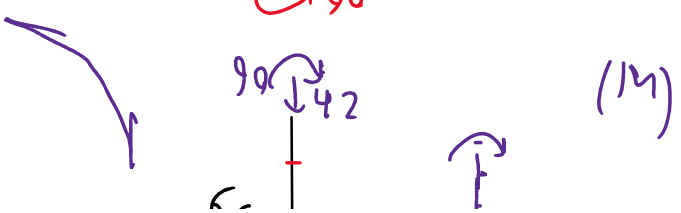
$3.4 - 2x = V \rightarrow x = 1.7\,m$

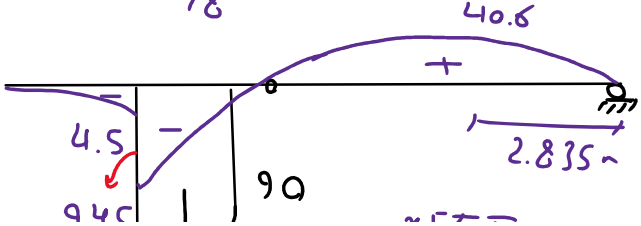
$M = 3.4 \times 1.7 - 2 \times \frac{1.7^2}{2} = 2.89$

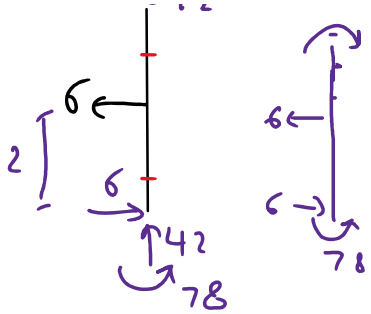
مثال :

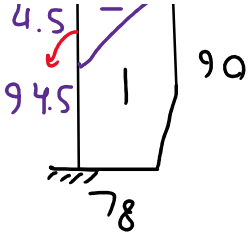




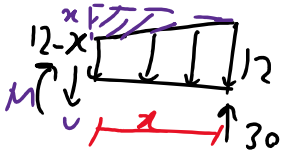




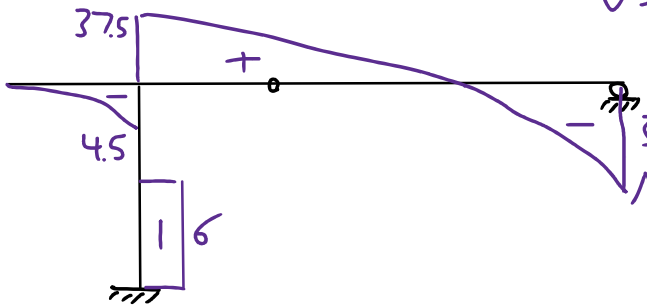




2.835



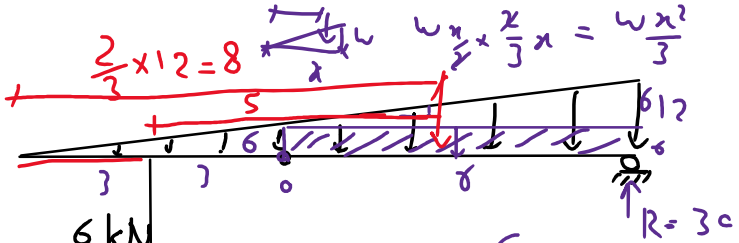
(V)

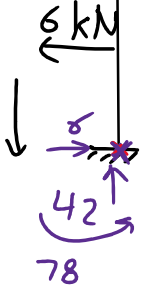


$$V = 30 - \int_0^x (12 - x)\,dx = 30 - \left(12x - \frac{x^2}{2}\right) = 0$$

$$\frac{x^2}{2} - 12x + 30 = 0 \rightarrow x = \frac{12 \pm 9.165}{1} = 2.835$$

$$M = 30x - 6x^2 + \frac{x^3}{6} = 40.6$$





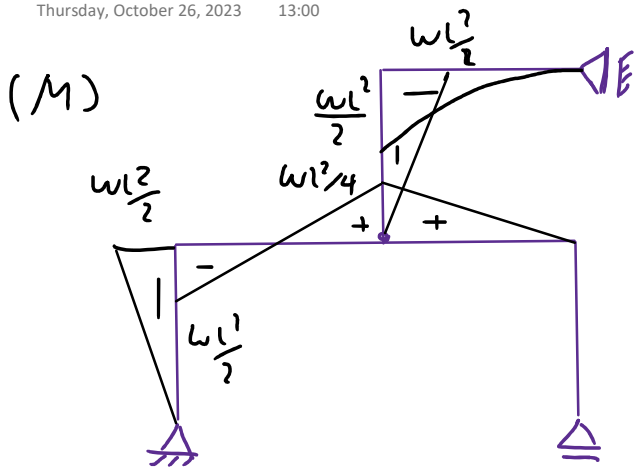
$$36 \times 3 + 6 \times \frac{6^2}{3} = 6R \rightarrow R = 30$$

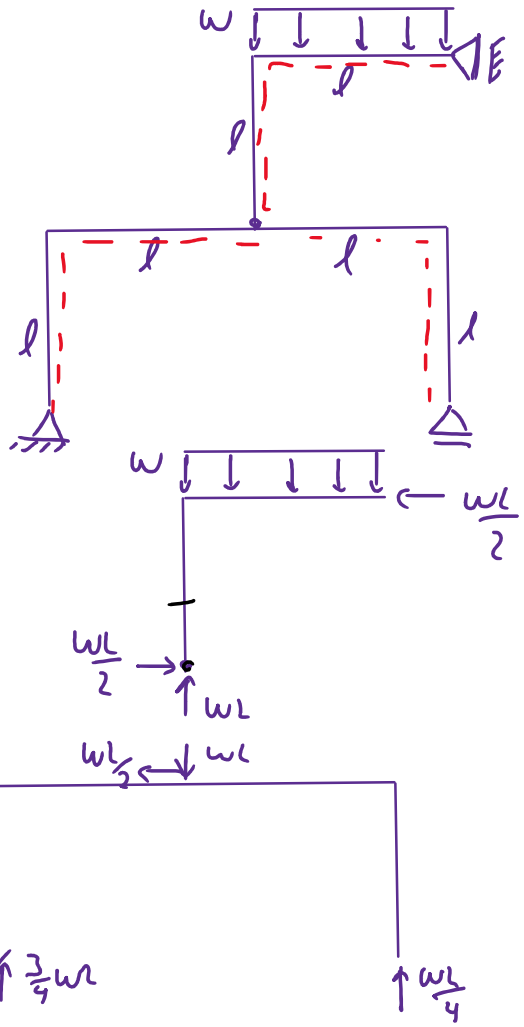
$$\frac{1}{2} \times 12 \times 12 - 30 = R$$

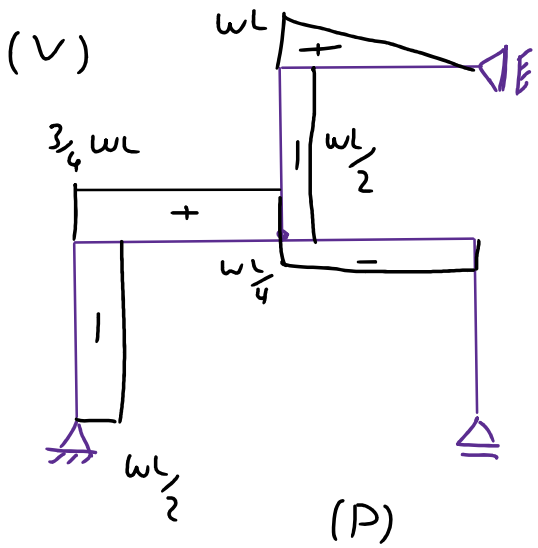
$$72 \times 5 - 30 \times 9 - 6 \times 2 =$$

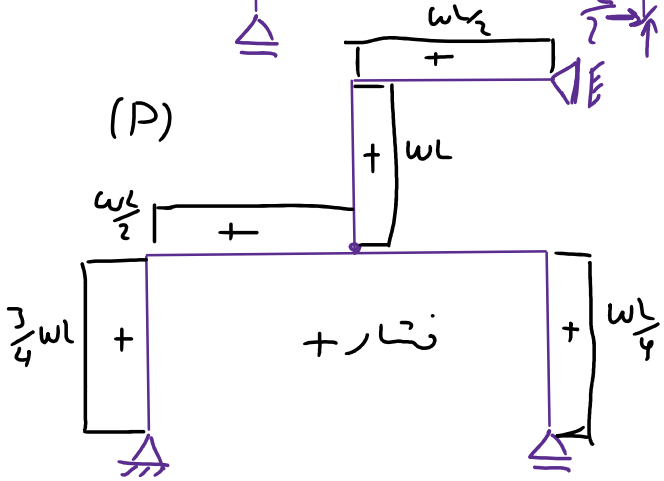
Frame2

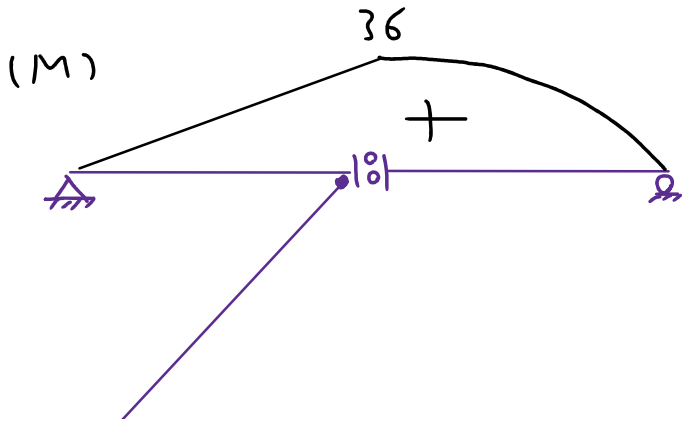
Thursday, October 26, 2023 13:00

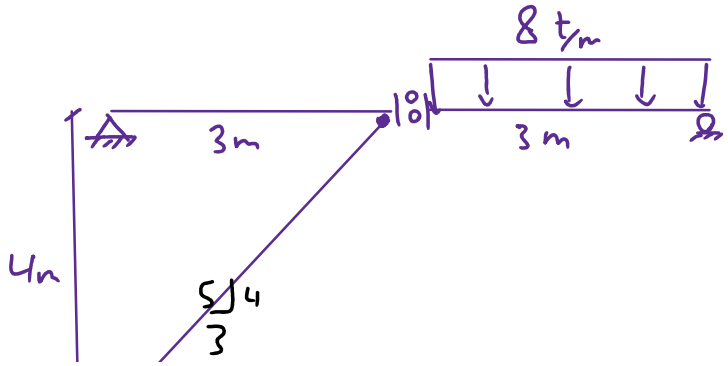


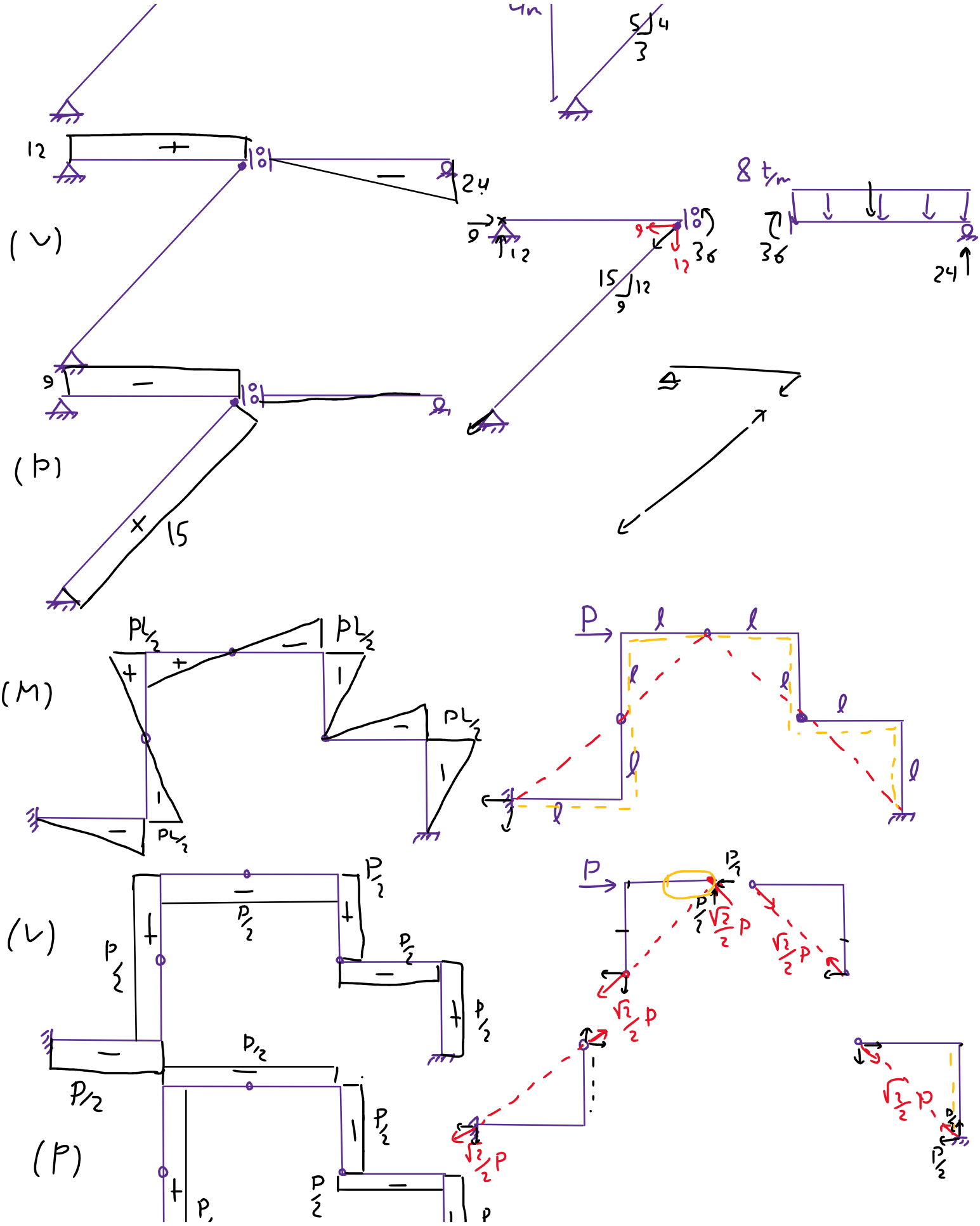











4m
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(V)
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12
36
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12
8 t/m
36
24
(b)
15
(M)
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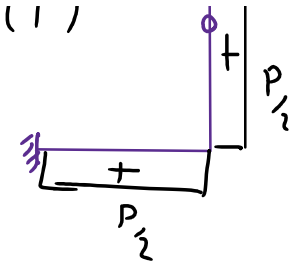
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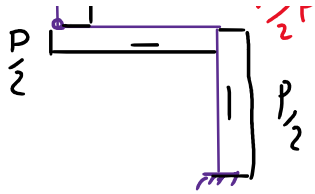
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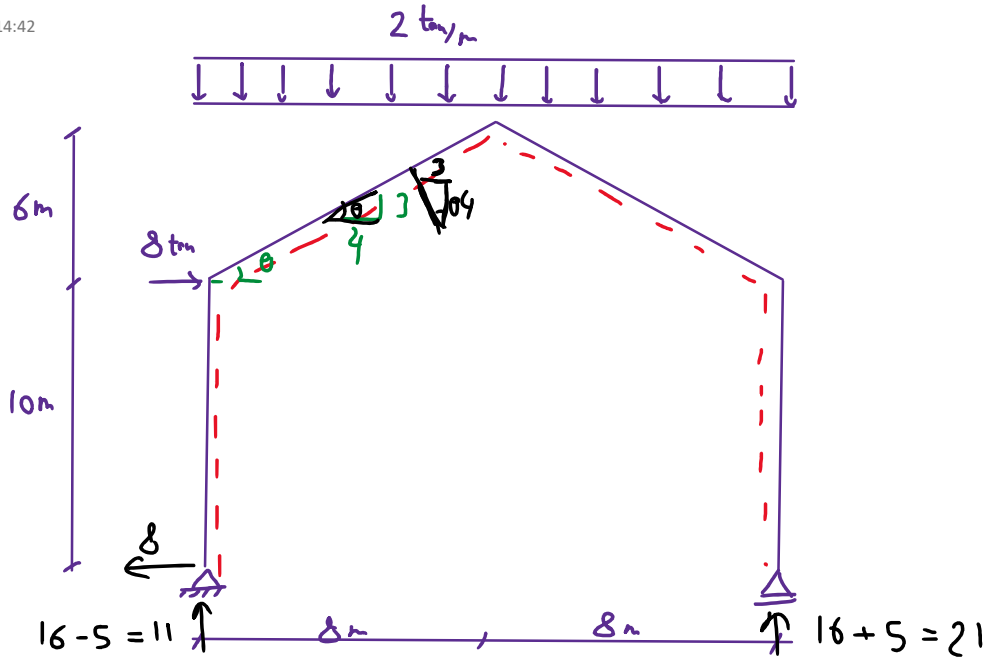
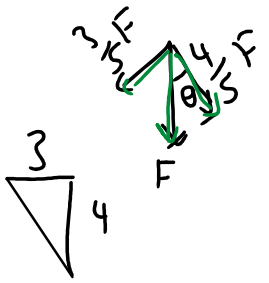
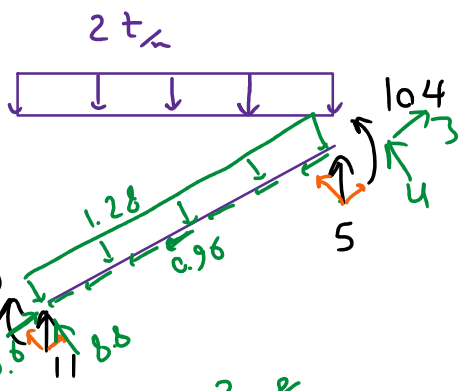
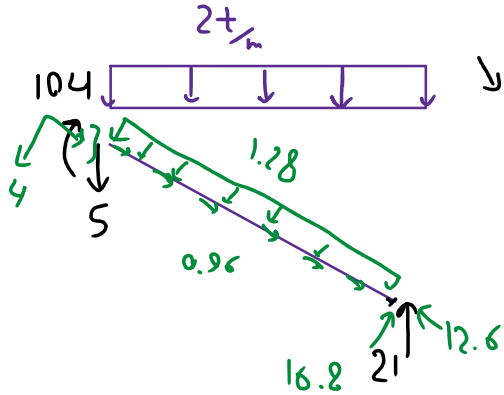
مثال:





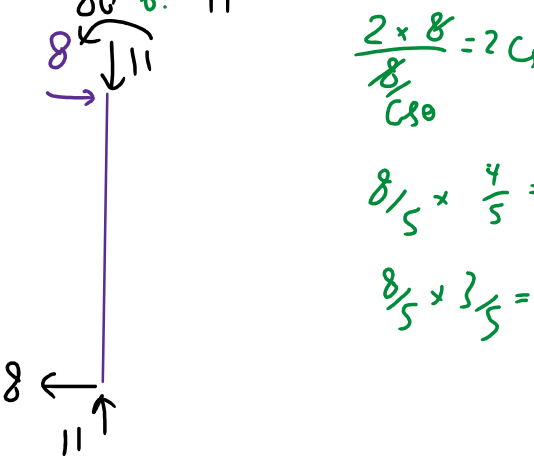


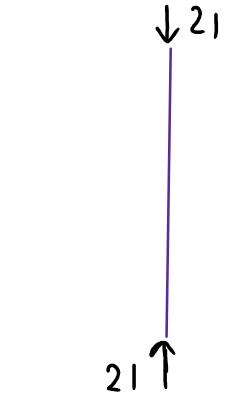


$\frac{2 \times 8}{8 \cos\theta} = 2 \sec\theta = 2 \times \frac{4}{5} = 1.6$

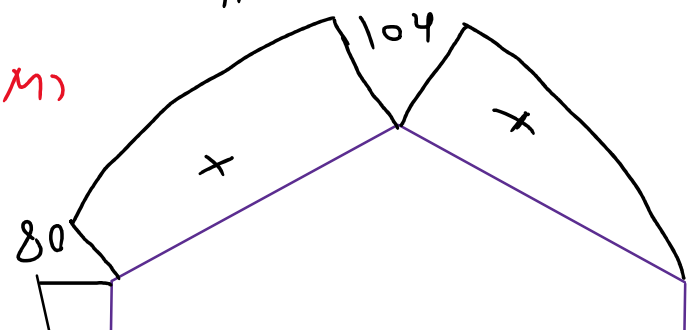
$\frac{8}{5} \times \frac{4}{5} = 1.28$

$\frac{8}{5} \times \frac{3}{5} = 0.96$

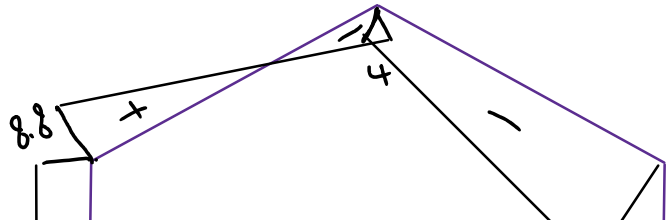


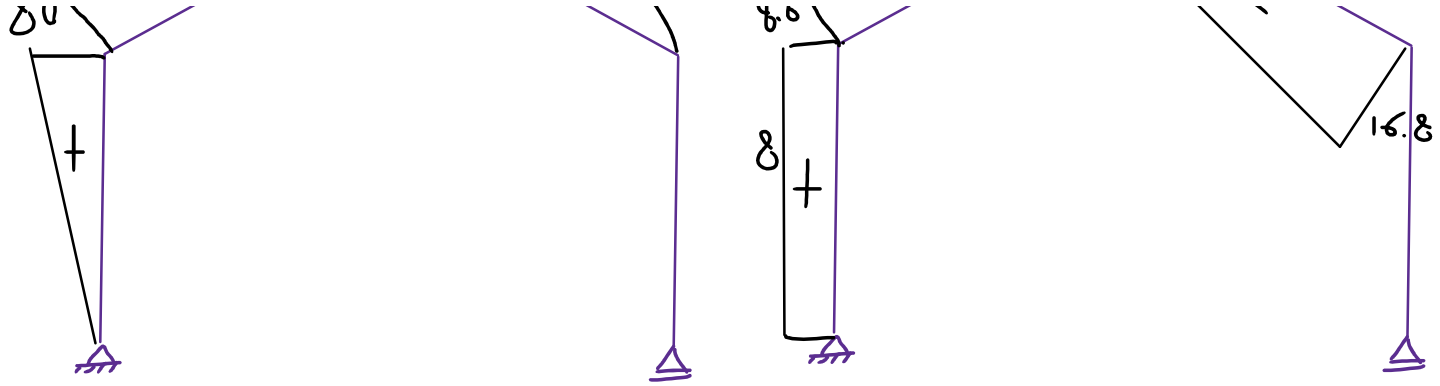


(M)

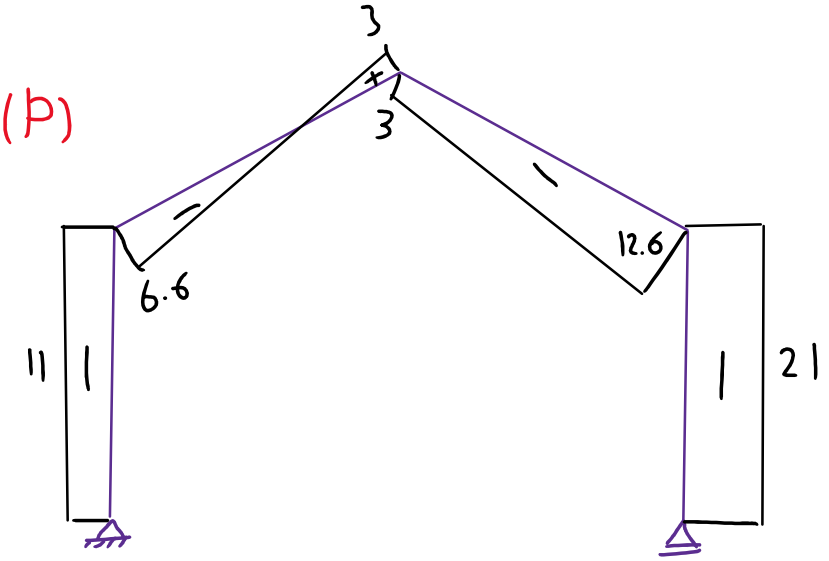


(V)





(b)



Truss 1

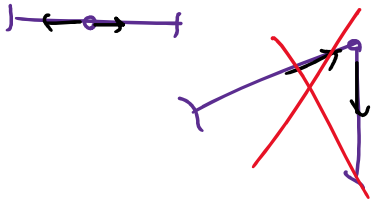
Tuesday, October 31, 2023 15:52

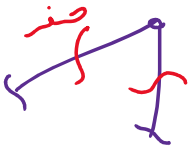
تحلیل خرپاهای معین

روش تحلیل خرپاهای معین:
1. روش گره
2. روش مقطع
3. روش عضو حذفی (روش هنبرگ)

ترکیبی

اعضای صفر نیرویی





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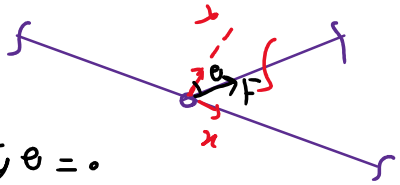
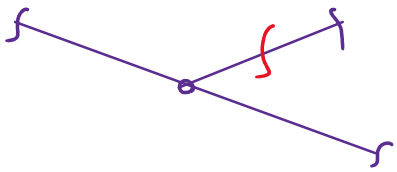


$F\cos\theta = 0$



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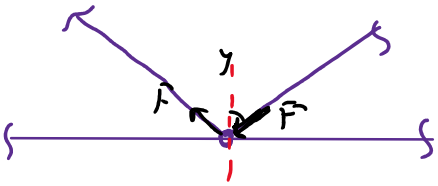
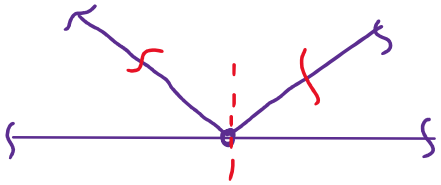


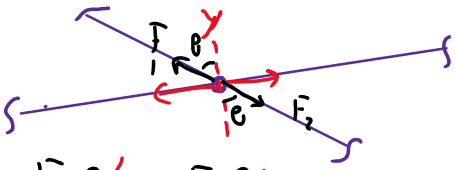
محور تقارن



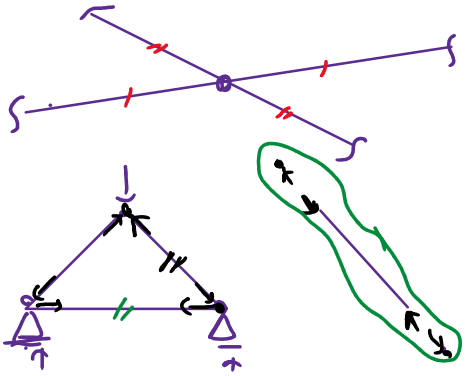
محور تقارن

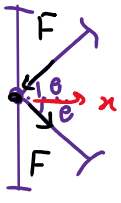
3

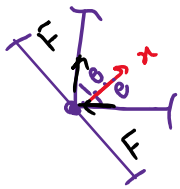
اعضای هم نیرو

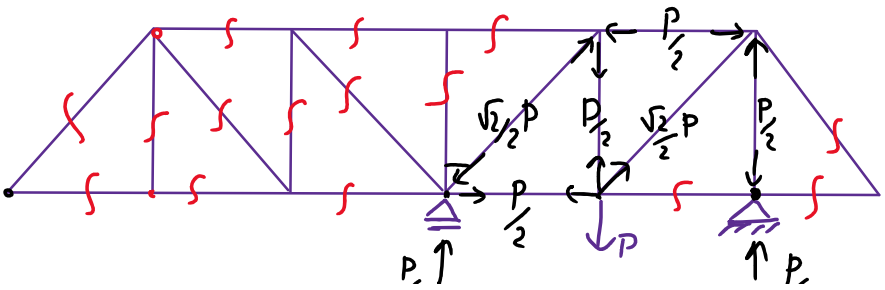


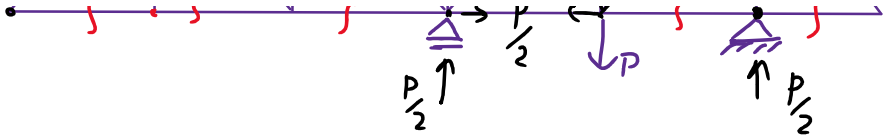
$F_1\cos\theta = F_2\cos\theta$







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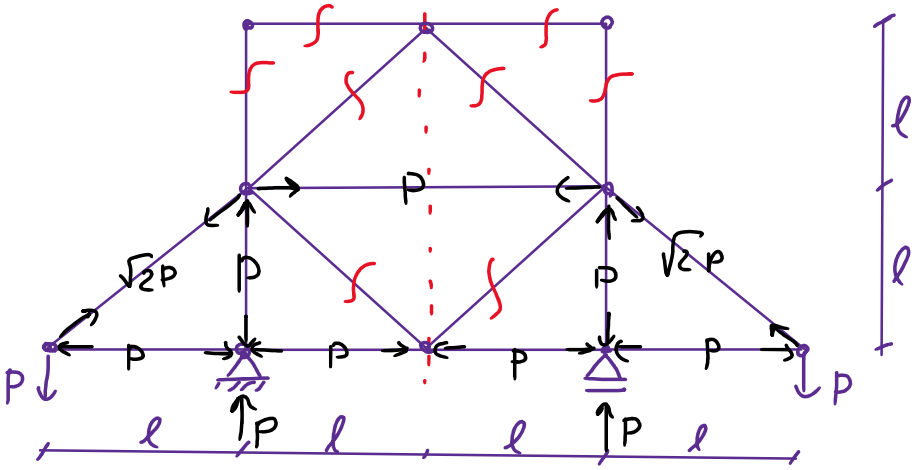
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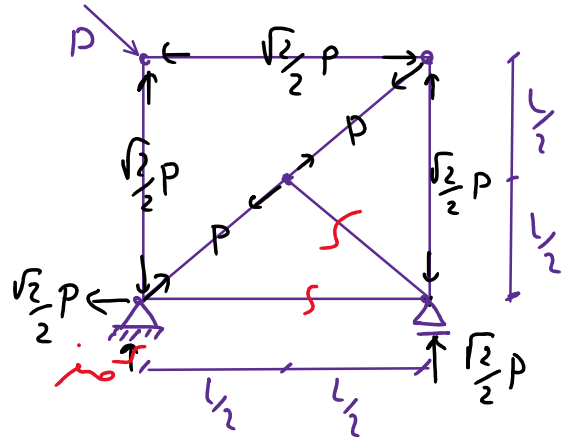
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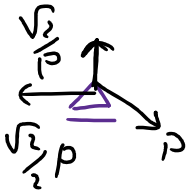
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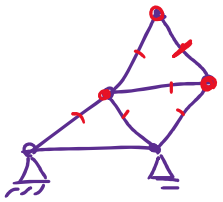
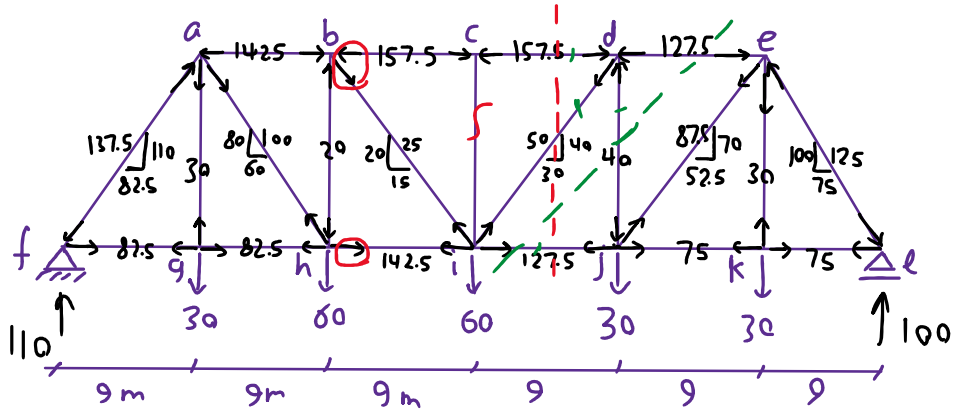


روش تعادل گره

در این روش، معادلات تعادل گره ($\sum F_x = 0$ و $\sum F_y = 0$) برای به دست آوردن دو مجهول در یک گره استفاده می‌شود. بنابراین باید در ابتدا از گره‌ای که دو مجهول دارد شروع کنیم و سپس با معلوم شدن سایر مجهولات گره‌های دیگر، دو مجهول باقی‌مانده را در آن گره‌ها را حساب کنیم.



روش مقطع

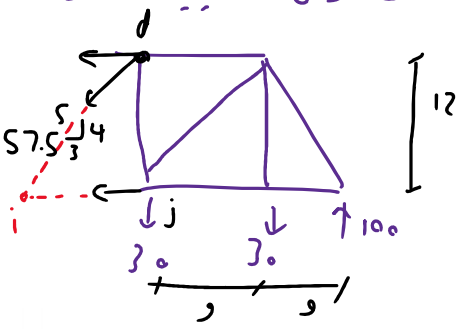
① برای به دست آوردن نیروی عضو خاصی در خرپای ساده (راحت‌تر از روش گره)

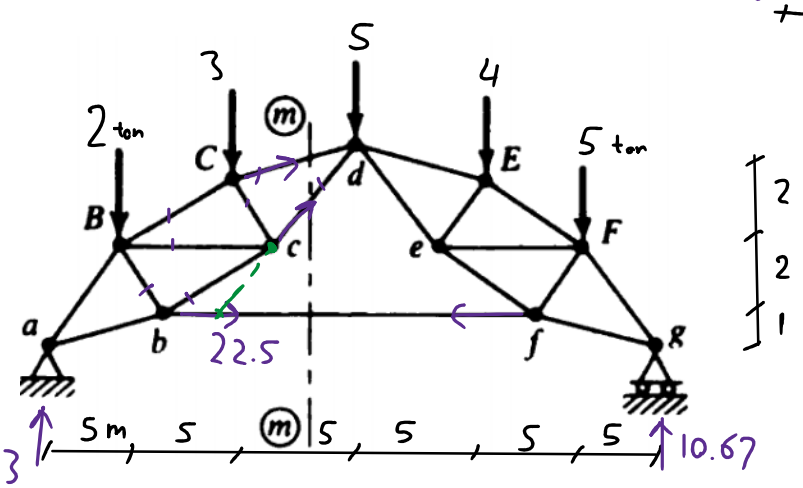
② وقتی آسان‌تر از روش گره وجود ندارد. (در خرپای مرکب یا پیچیده لازم است)

$\sum M_d = 0 \rightarrow F_{ij}(12) + 30(9) - 100(18) = 0 \rightarrow F_{ij} = 127.5$

$\sum M_i = 0 \rightarrow -F_{cd}(12) + 30(9) + 30(18) - 100(27) = 0 \rightarrow F_{cd} = -157.5$

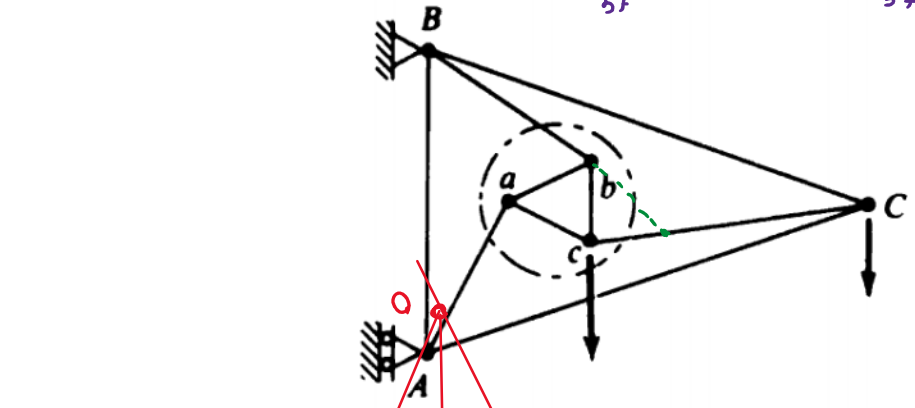
$\sum F_y = 0 \rightarrow -\frac{4}{5} F_{di} - 30 - 30 + 100 = 0 \rightarrow F_{di} = 50$

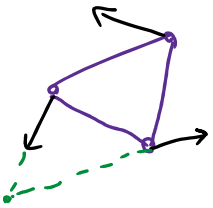


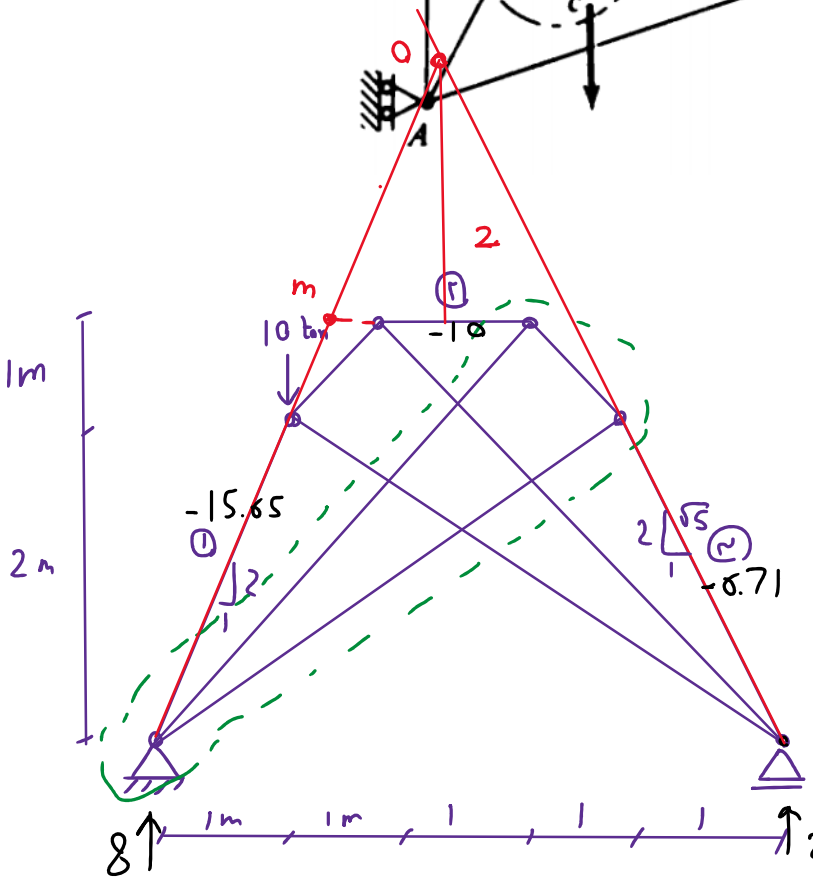


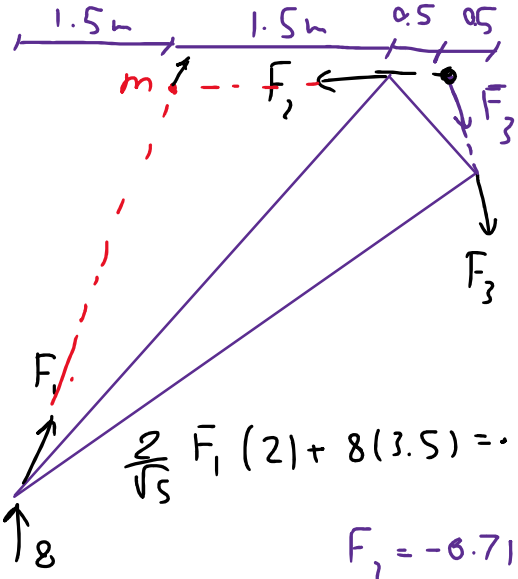
$\sum M_d = 0 \rightarrow 8.33 \times 15 - 2 \times 10 - 3 \times 5 - F_{bf} \times 4 = 0 \rightarrow F_{bf} = 22.5$

B









$\frac{2}{\sqrt{5}} F_1 (2) + 8(3.5) = 0$

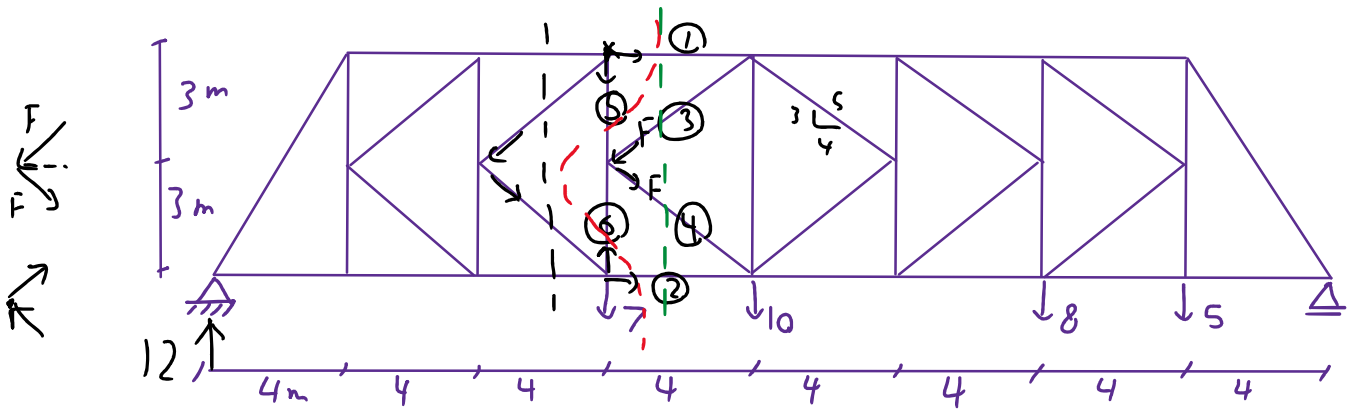
$F_3 = -6.71$

$\sum M_m = 0 \quad 8(1.5) + \frac{2}{\sqrt{5}} F_3 (2) = 0$ ↻

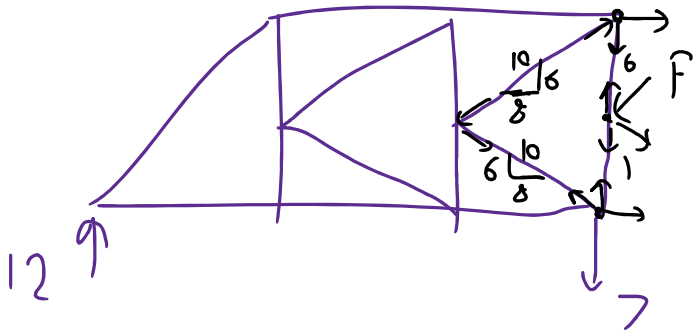
$\sum M_o = 0 \rightarrow 8(2.5) + F_2(2) = 0 \rightarrow F_2 = -10$

$\sum F_y = 0 \quad \frac{2}{\sqrt{5}} F_1 + 8 + \frac{2}{\sqrt{5}} \times 6.71 = 0$

$F_1 = -15.65$



$12 \times 12 - F(8) = 0 \rightarrow F = 24$



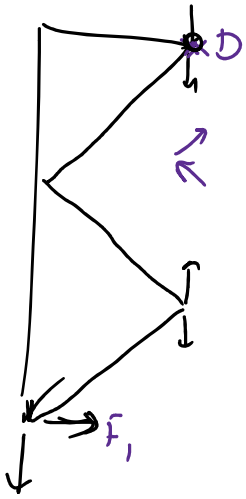
$12\times12 - F_2(8) = 0 \rightarrow F_2 = 24$

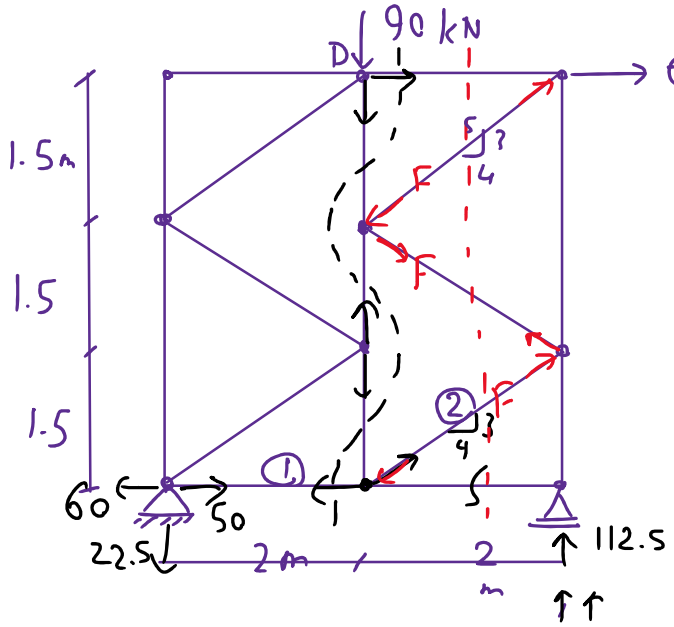
$\Sigma F_x = 0 \rightarrow F_1 = -24$

$\Sigma F_y = 0$

$2\times\frac{3}{5}F - 12 + 7 = 0 \rightarrow F_3 = F_4 = 4.16$ (فشاری)

$F_5 = 6 \qquad F_6 = 1$





$\Sigma M_D = 0$

$22.5(2) - 60(4.5) + F_1(4.5) = 0$

$F_1 = 50$

$\frac{4}{5}F = 50 \rightarrow F_2 = 62.5$

$\Sigma F_y = 0$

$112.5 + 3\times\frac{3}{5}F = 0 \rightarrow F_2 = -62.5$

$45 + 60\times\frac{4.5}{4} =$

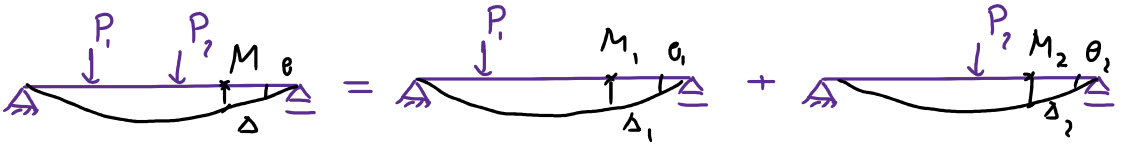
Truss

Thursday, November 2, 2023 12:36

روش عضو جایگزین (روش هنبرگ)

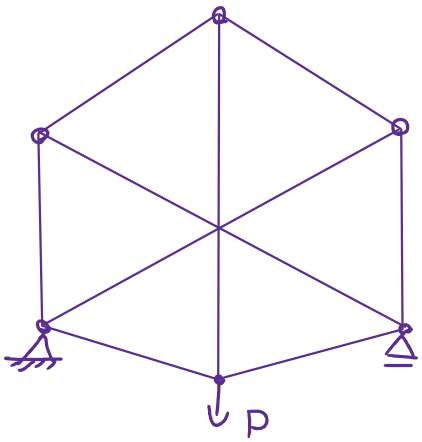
روش عضو جایگزین در سال ۱۸۸۶ برای تحلیل خرپا توسط هنبرگ ارائه شد. این روش بر اصل جمع آثار قوا (اصل سوپرپوزیشن) استوار است.

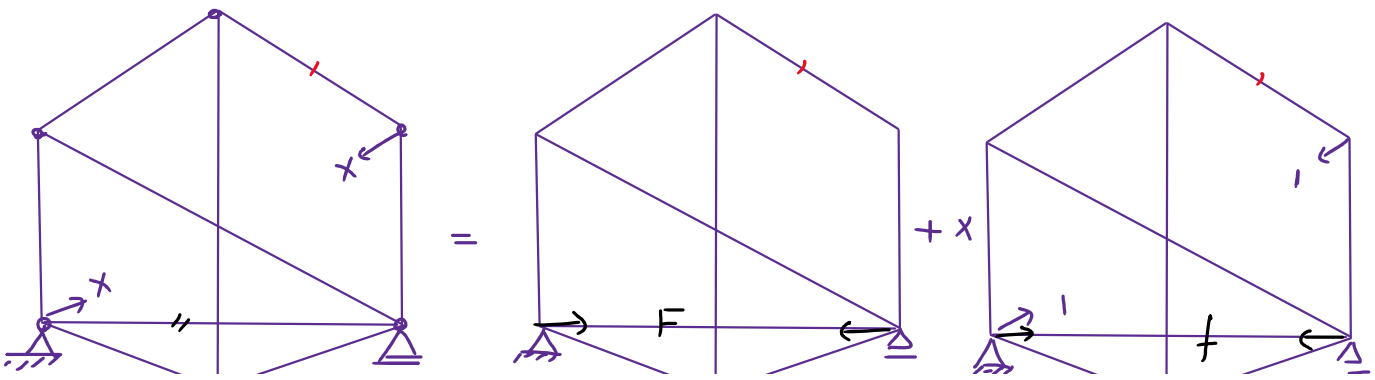
اصل جمع آثار قوا (سوپرپوزیشن)

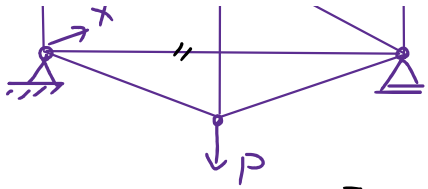


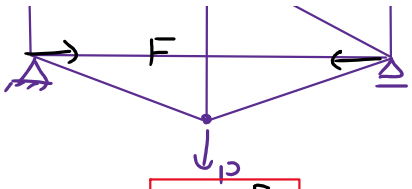
$$\begin{cases} M = M_1 + M_2 \\ \Delta = \Delta_1 + \Delta_2 \\ v = v_1 + v_2 \\ \vdots \end{cases}$$

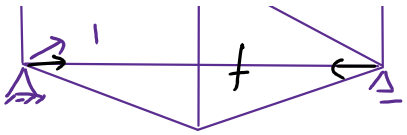
* فقط در محدوده خطی صحیح است.





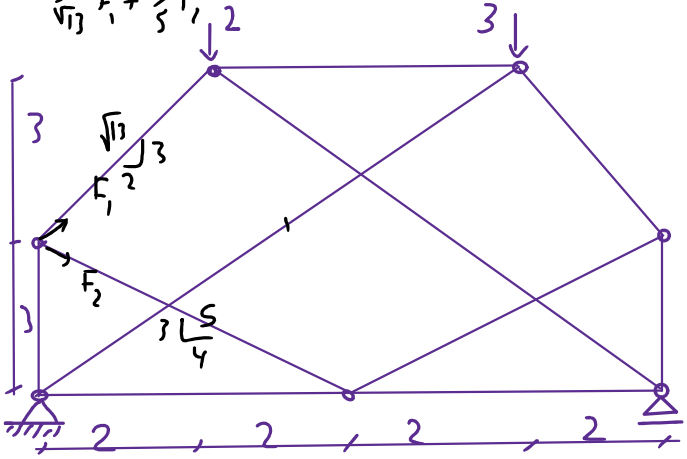


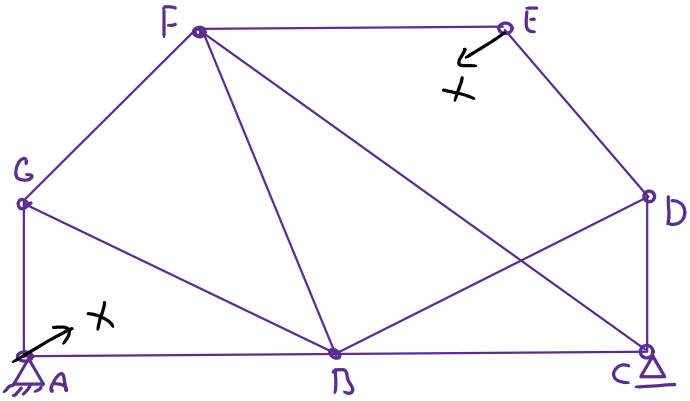


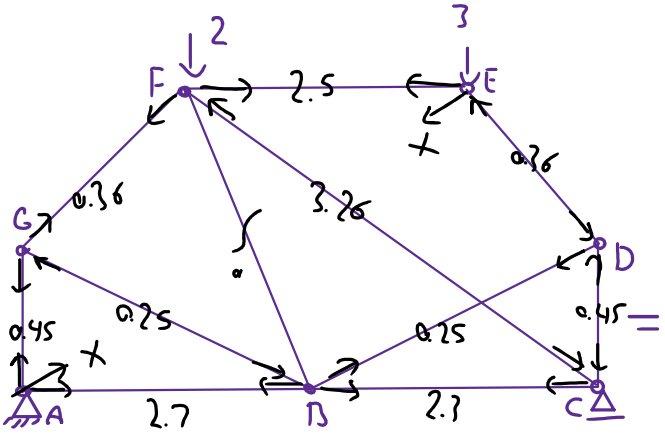


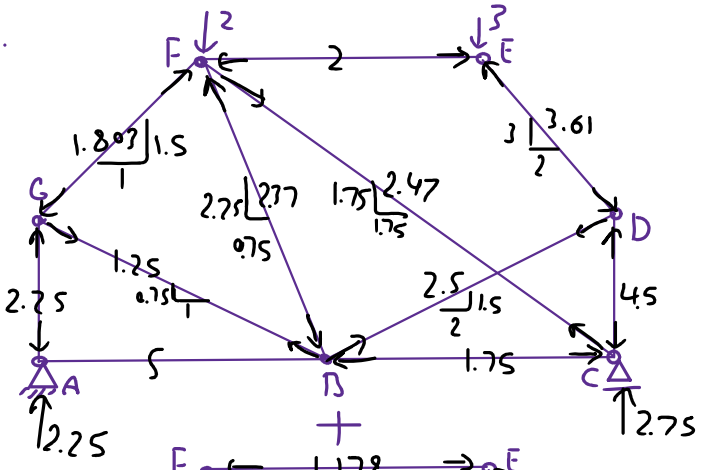
$$F + X f = 0 \longrightarrow \boxed{X = -\frac{F}{f}}$$

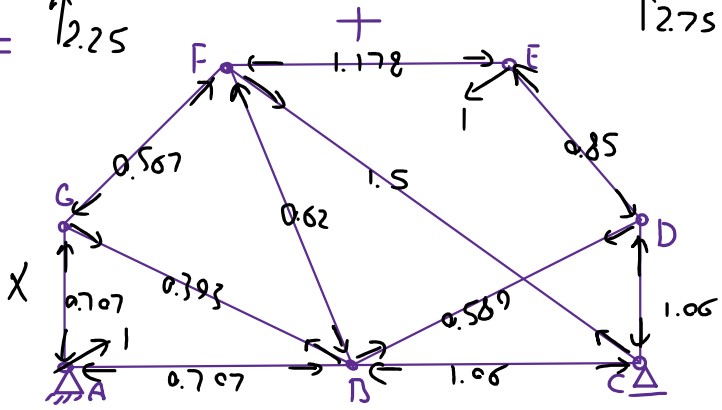
مثال :





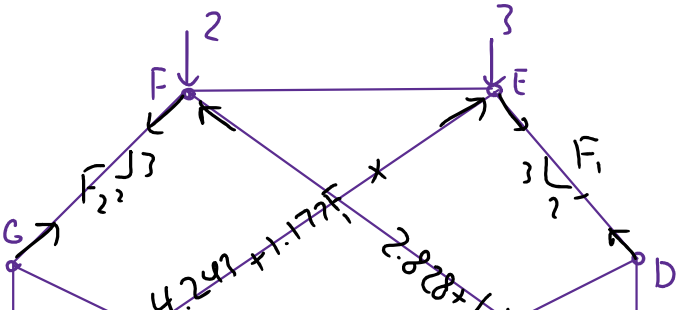


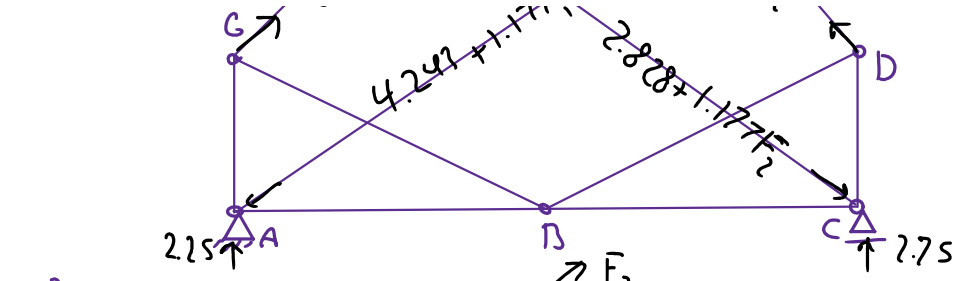




$$-2.37 + X(-0.62) = 0$$

$$\boxed{X = -3.823}$$

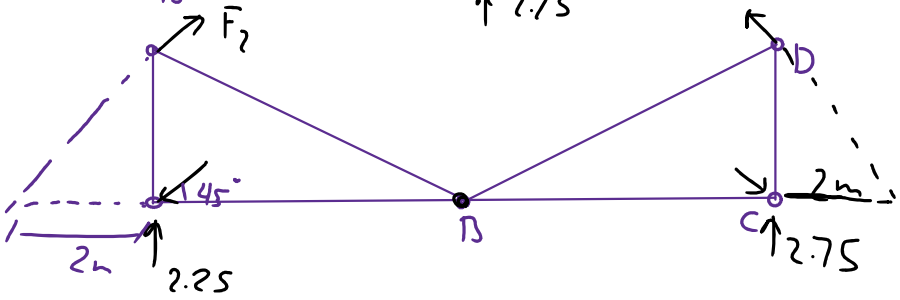




$$\frac{\sqrt{2}}{2}x - \frac{3}{\sqrt{13}}F_1 - \frac{3}{2} = 0$$

$$x = 4.243 + 1.177F_1$$

2.828



$$* \quad 2.75 \times 4 - \frac{\sqrt{2}}{2}(2.828 + 1.177F_2) \times 4 + \frac{3}{\sqrt{13}}F_1 \times 6 = 0$$

$\Sigma M_B =$

$$2.25 \times 4 - \frac{\sqrt{2}}{2}(4.243 + 1.177F_1) \times 4 + \frac{3}{\sqrt{13}}F_2 \times 6 = 0$$

3.329 | $4.992F_2 - 3.329F_1 = 3.001$

4.992 | $-3.329F_2 + 4.992F_1 = -3.001$

$\rightarrow \quad (-3.329^2 + 4.992^2)F_1 =$

$3.001(3.329 - 4.992) \rightarrow \boxed{F_1 = -0.361}$

$F_2 = 0.361$

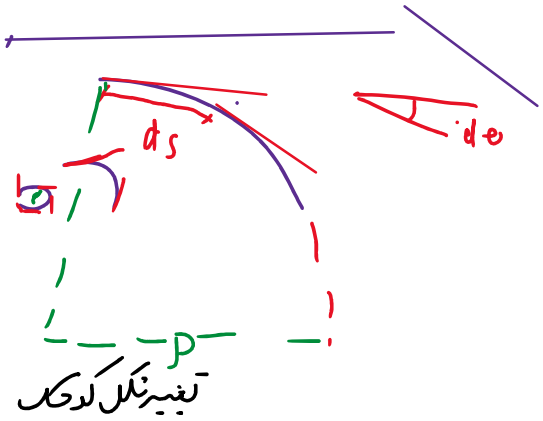
Deflections1

Monday, November 6, 2023 11:43

فصل سوم: محاسبه تغییر شکل های ناشی از خمش به روش های غیر انرژی

انتگرال مضاعف (مقاومت مصالح)

① تیر طره
② تیر دوسر



تغییر شکل کوچک

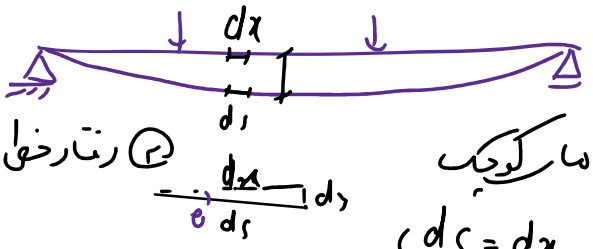
رابطه لنگر - انحنا (moment - Curvature)

تعریف هندسی

$$\text{انحنا } \phi = \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{dy'}{dx} = y'' \quad (1)$$

تعریف مقاومت مصالح

$$\text{انحنا } \phi = \frac{1}{\rho} = \frac{\varepsilon}{y} = \frac{\sigma}{Ey} = \frac{My/I}{Ey} = \frac{M}{EI} \quad (2)$$



① تغییر شکل های کوچک

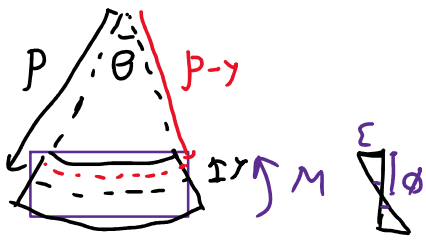
$$\begin{cases} ds = dx \\ \theta = \tan\theta = y' \end{cases}$$

$$\boxed{y'' = \frac{M}{EI}}$$

$$\varepsilon = \frac{\Delta L}{L} = \frac{(\rho - y)\theta - \rho\theta}{\rho\theta}$$

$$\varepsilon = -\frac{y}{\rho}$$

$$\phi = \frac{1}{\rho} = \frac{\varepsilon}{y} \qquad \sigma = \frac{My}{I}$$



$$y'' = \frac{M}{EI}$$

$$\frac{dy}{dx} = w \qquad \frac{dM}{dx} = V$$

$$M = \int \sigma y\, dA = \int \frac{y}{c}\sigma_{max}\, y\, dA$$

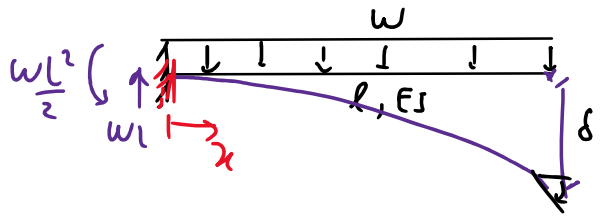
$$M = \frac{\sigma_{max}}{c}\int y^2 dA \quad I$$

* $\theta = y' = \int \frac{M}{EI}\, dx$
* $y = \int \theta\, dx$

$$V = \int w\, dx$$

$$M = \int V\, dx$$

مثال:



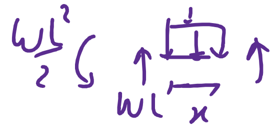
$$\theta = \int_0^x \frac{M}{EI}\, dx = \frac{1}{EI}\int_0^x M\, dx$$

$$\theta = \frac{1}{EI}\left(-\frac{wL^2}{2}x + \frac{wL}{2}x^2 - \frac{w}{6}x^3 + C_1\right)$$

$wL^2/2$

$$\theta = \frac{1}{EI}\left(-\frac{wL^2}{2}x + \frac{wL}{2}x^2 - \frac{w}{6}x^3 + C_1\right)$$

$x = 0 \rightarrow \theta = 0$



$$M(x) = -\frac{wL^2}{2} + wLx - w\frac{x^2}{2}$$

$$y = \int \theta \, dx$$

$$y = \frac{1}{EI}\left(-\frac{wL^2}{4}x^2 + \frac{wL}{6}x^3 - \frac{w}{24}x^4 + C_2\right)$$

$x = 0 \rightarrow y = 0$ $\qquad \frac{-6+4-1}{24} = \frac{-3}{24}$

$$\theta = \frac{-wL^3}{6EI} \qquad \delta = -\frac{wL^4}{8EI}$$

روش لنگر-سطح moment-area

فرمول بندی توسط چارلز گرین در ۱۸۷۳ میلادی

$$y'' = \frac{M}{EI} \rightarrow \frac{dy'}{dx} = \frac{d\theta}{dx} = \frac{M}{EI}$$

$$d\theta = \frac{M}{EI}dx$$

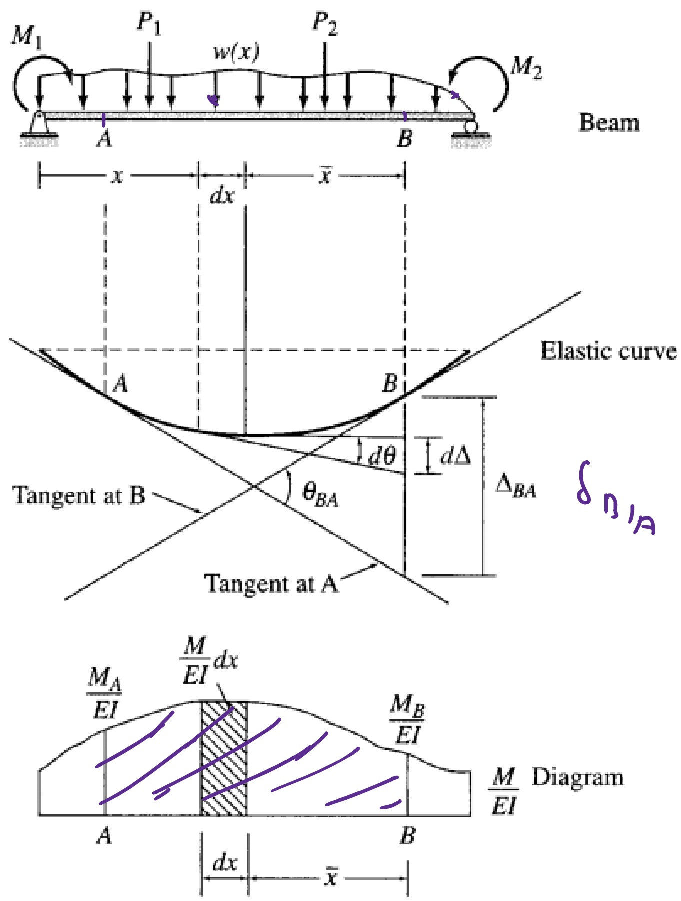
$$\int_{\theta_A}^{\theta_B} d\theta = \int_{x_A}^{x_B} \frac{M}{EI}dx$$

$$\theta_{B/A} = \int_{x_A}^{x_B} \frac{M}{EI}dx = S$$

قضیه اول

$\theta_{B/A}$: چرخش نقطه B نسبت به نقطه A (براساس جهت مثبت مثلثاتی)

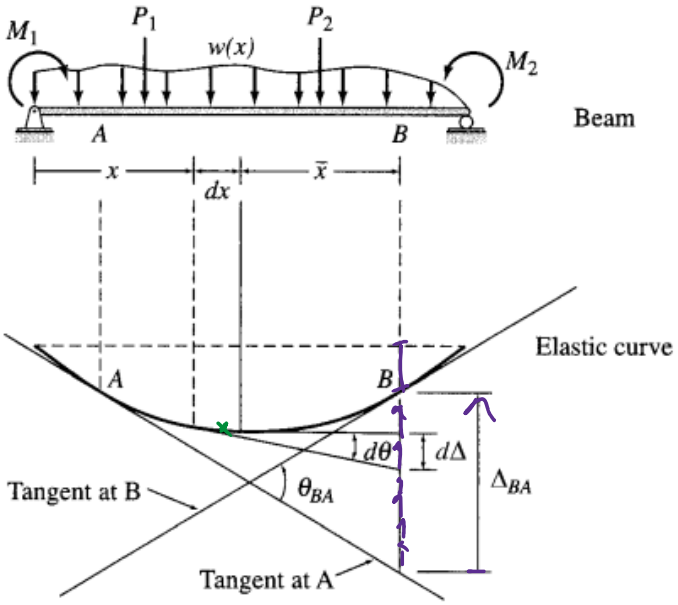
S: مساحت زیر نمودار $\frac{M}{EI}$



$$d\delta = \bar{x}\, d\theta = \bar{x}\,\frac{M}{EI}\,dx$$

$$\delta_{B/A} = \int_{x_A}^{x_B} \bar{x}\,\frac{M}{EI}\,dx = S\,\bar{X}$$

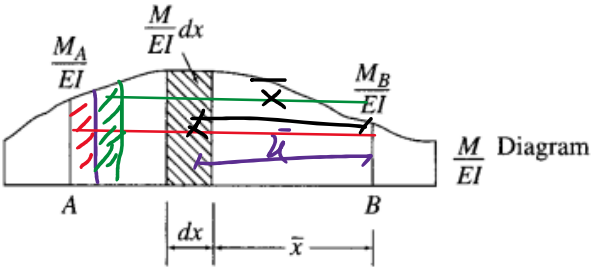
قضیه دوم

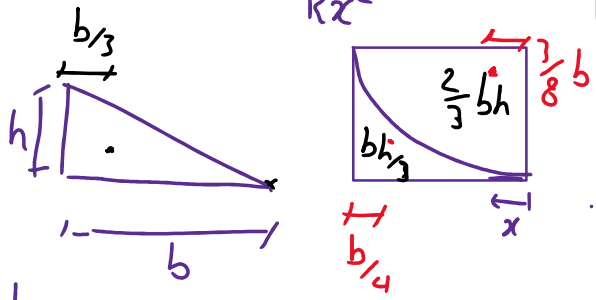


$\delta_{B/A}$: انحراف نقطه B از مماس بر A (براساس بالا) (مثبت)

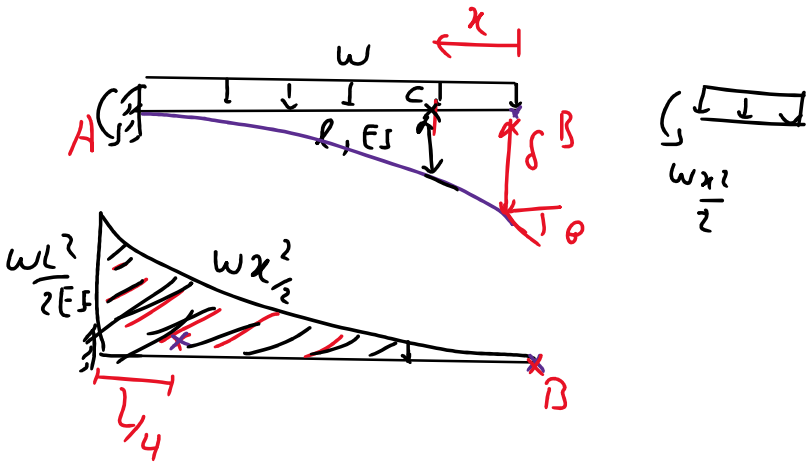
S: مساحت زیر نمودار $\frac{M}{EI}$

$\bar{X}$: فاصله مرکز سطح (S) تا B





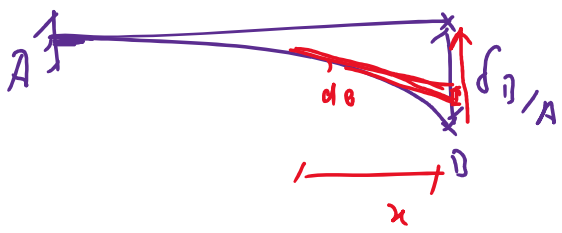
$$S = \frac{bh}{2}$$



$$\theta_{B/A} = \theta_B - \theta_A = S =$$

$$\frac{1}{3}\left(\frac{wL^2}{2EI}\right)L$$

$$\theta_B = \frac{wL^3}{6EI}$$



$$\delta_{B/A} = \delta_B = \left(\frac{wL^3}{6EI}\right)\left(\frac{3}{4}L\right)$$

$$\delta_B = \frac{wL^4}{8EI}$$

$$\delta_B = \frac{wL^4}{8EI}$$

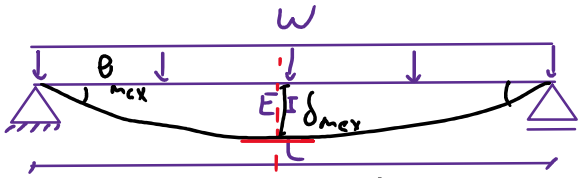
Deflections2

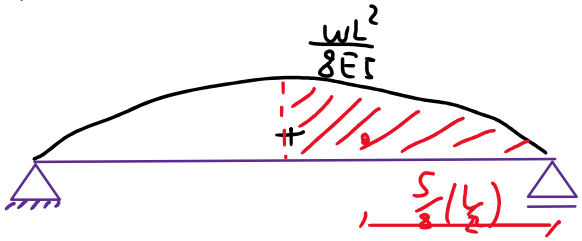
Tuesday, November 7, 2023

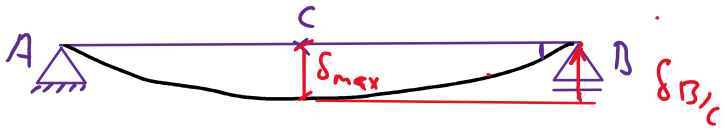
مثال: δ_{max} , θ_{max}

$$\theta = \theta_{max} = \theta_{B/C} = \theta_B - \theta_C = \frac{2}{3}\left(\frac{wL^2}{8EI}\right)\left(\frac{L}{2}\right) = \frac{wL^3}{24EI}$$

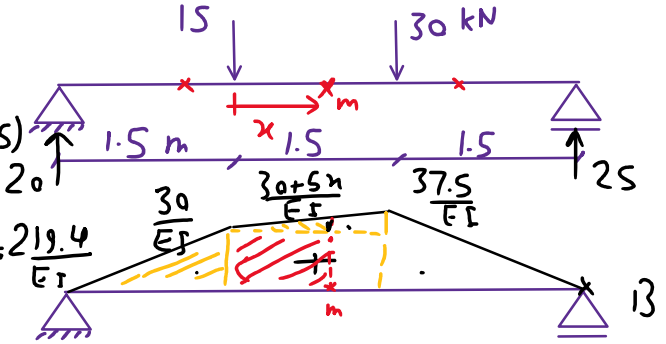
$$\delta_{max} = \delta_{B/C} = \frac{wL^3}{24EI}\left(\frac{5}{8}\times\frac{L}{2}\right) = \frac{5wL^4}{384EI}$$







مثال: $\delta_{max} = ?$

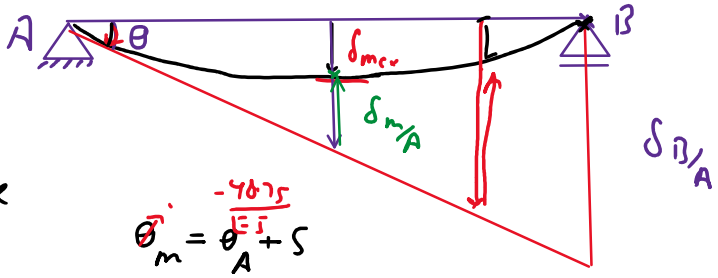


$$\delta_{B/A} = \frac{1}{2}\left(\frac{30}{EI}\right)(1.5)(3.5) + \left(\frac{30}{EI}\right)(1.5)(2.25) + \frac{1}{2}\left(\frac{7.5}{EI}\right)(1.5)(2) + \frac{1}{2}\left(\frac{37.5}{EI}\right)(1.5)(1) = \frac{219.4}{EI}$$

$$\theta_A = \frac{\delta_{B/A}}{L} = \frac{48.75}{EI}$$

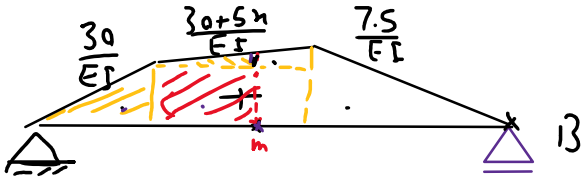
$$\theta_{m/A} = \theta_m - \theta_A = \frac{1}{2}\left(\frac{30}{EI}\right)(1.5) + \frac{1}{2}\left[\frac{30}{EI} + \frac{30+5x}{EI}\right)x$$

($\theta_A = -\frac{48.75}{EI}$)



$$\theta_m = \theta_A + S \quad \left(-\frac{48.75}{EI}\right)$$

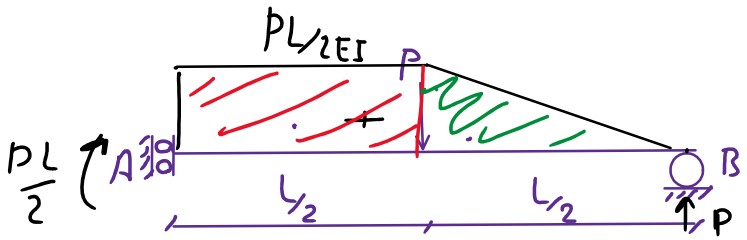
$$-26.25 + 30x + 2.5x^2 = 0 \rightarrow \boxed{x = 0.819\ m}$$



$$\delta_{max} = L\theta_A + \delta_{m/A} = 2.319 \times \frac{-48.75}{EI} + \frac{1}{2}\left(\frac{30}{EI}\right)(1.5)(1.319) + \left(\frac{30}{EI}\right)\left(\frac{0.819^2}{2}\right) + \frac{1}{2}\left(\frac{5\times 0.819}{EI}\right)(0.819^2) = -72.85$$

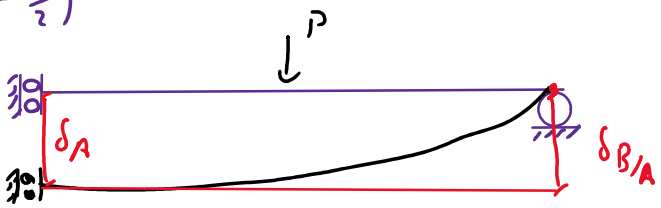
$$+\frac{1}{2}\left(\frac{5\times 0.819}{EI}\right)\left(0.819^2\right)=\frac{72.85}{EI}$$

مثال: $\delta_A=?$



$$\delta_A=\delta_{B/A}=$$

$$\left(\frac{PL}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{3L}{4}\right)+\frac{1}{2}\left(\frac{PL}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{2}{3}\frac{L}{2}\right)$$



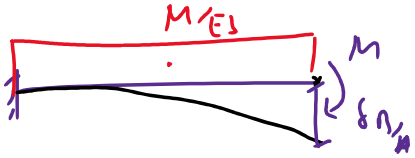
$$\frac{3\times 3+2}{48}$$

$$\boxed{\delta_A=\frac{11}{48}\frac{PL^3}{EI}}$$

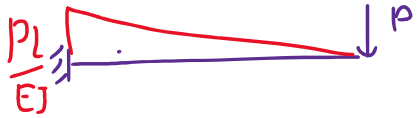
EI : $kg/cm^2 \cdot cm^4 = kg\cdot cm^2$ ، $\frac{kg\cdot m^3}{kg\cdot cm^2}=cm$

$$\theta=\frac{PL^2}{EI}\qquad \frac{wL^4}{EI}\qquad k_s$$

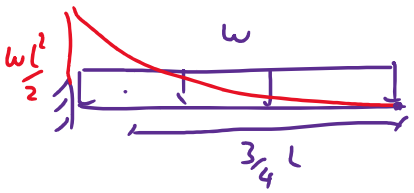
$$\theta=\frac{wL^3}{EI}$$



$$\theta=\frac{ML}{EI}\qquad \delta=\frac{ML^2}{2EI}$$



$$\theta=\frac{PL^2}{2EI}\qquad \delta=\frac{PL^3}{3EI}$$

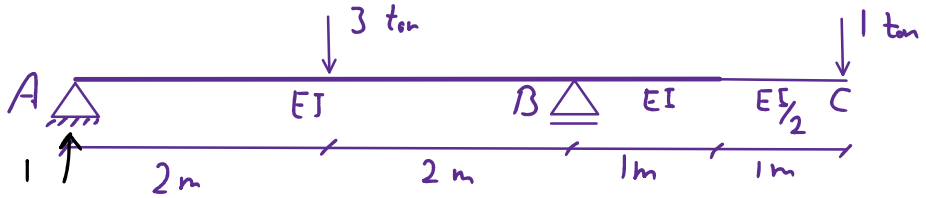


$$\theta=\frac{wL^3}{6EI}\qquad \delta=\frac{wL^4}{8EI}$$

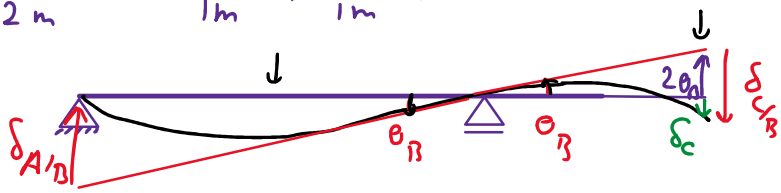
Deflection3

Thursday, November 9, 2023 9:59

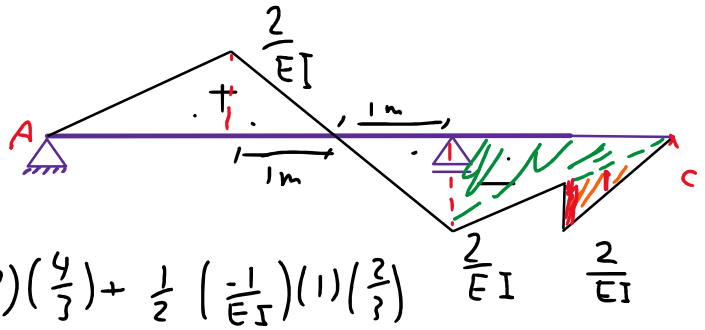
$\delta_c = ?$: مثال



$$\delta_{A/B} = \frac{1}{2}\left(\frac{2}{EI}\right)(2)\left(\frac{4}{3}\right) + \frac{1}{2}\left(\frac{2}{EI}\right)(1)\left(\frac{7}{3}\right) - \frac{1}{2}\left(\frac{2}{EI}\right)(1)\left(\frac{11}{3}\right) = \frac{4}{3EI}$$



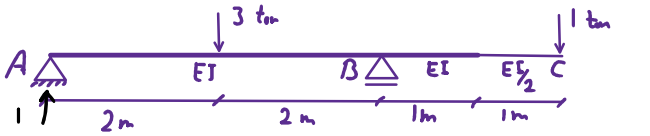
$$\theta_B = \frac{\frac{4}{3EI}}{4} = \frac{1}{3EI}$$



$$\delta_c = 2\theta_B + \delta_{C/B} = 2\left(\frac{1}{3EI}\right) + \frac{1}{2}\left(\frac{-2}{EI}\right)(2)\left(\frac{4}{3}\right) + \frac{1}{2}\left(\frac{-1}{EI}\right)(1)\left(\frac{2}{3}\right)$$

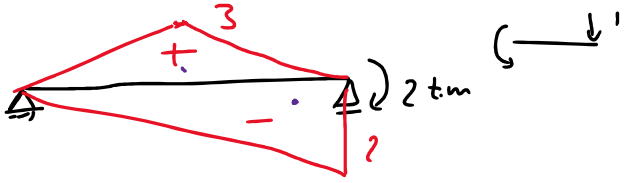
$$\frac{2}{3EI} - \frac{3}{EI}$$

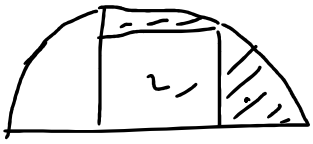
$$\boxed{\delta_c = \frac{-7}{3EI}}$$



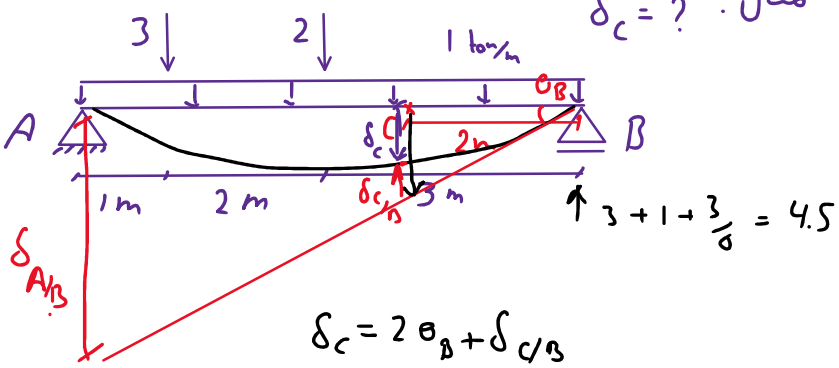
$$\delta_{A/B} = \frac{1}{2}\left(\frac{3}{EI}\right)(4)(2) + \frac{1}{2}\left(\frac{-2}{EI}\right)(4)\left(\frac{8}{3}\right)$$

$$= 12 + \left(\frac{-32}{3}\right) = \frac{36-32}{3} = \frac{4}{3EI}$$





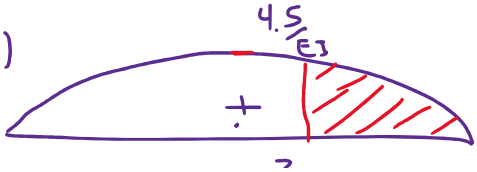
$\delta_c = ?$: مثال



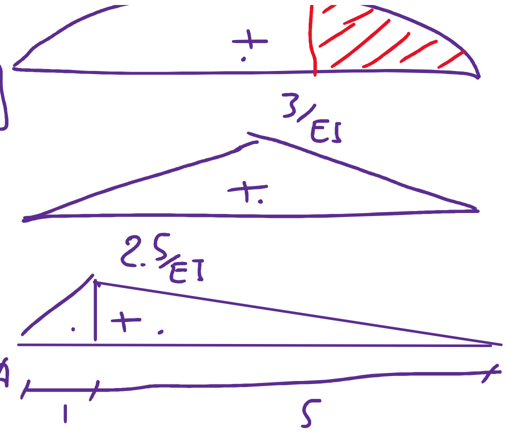
$$\delta_c = 2\theta_B + \delta_{C/B}$$

$$\delta_{A/B} = \frac{2}{3}\left(\frac{4.5}{EI}\right)(6)(3) + \frac{1}{2}\left(\frac{3}{EI}\right)(6)(3)$$

$$+\left[\frac{1}{2}(2.5)(1)\left(\frac{1}{3}\right) + \frac{1}{2}(2.5)(6)\left(\frac{8}{3}\right)\right]$$



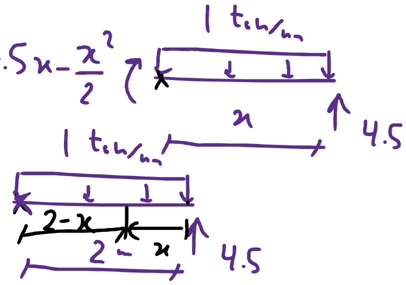
$$+\left[\frac{1}{2}\left(\frac{2.5}{EI}\right)(1)\left(\frac{2}{3}\right)+\frac{1}{2}\left(\frac{2.5}{EI}\right)(5)\left(\frac{8}{3}\right)\right]$$



$$\delta_{A/B}=\frac{98.5}{EI}$$

$$\theta_B=\frac{\delta_{A/B}}{6}=\frac{16.42}{EI}$$

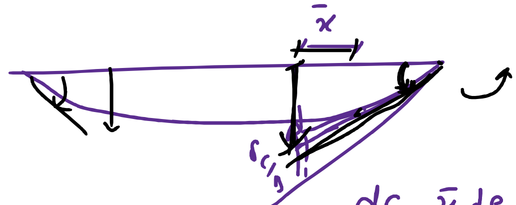
$$\delta_{C/B}=\int_0^2\frac{M}{EI}\bar{x}\,dx=\frac{1}{EI}\int_0^2\left(4.5x-\frac{x^2}{2}\right)(2-x)\,dx$$



$$\int_0^2\left(9x-4.5x^2-x^2+\frac{x^3}{2}\right)dx=$$

$$\frac{9}{2}x^2-\frac{5.5}{3}x^3+\frac{x^4}{8}\Big]_0^2=\frac{5.33}{EI}$$

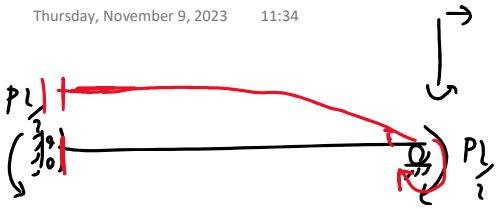
$$\delta_C=-2\left(\frac{16.42}{EI}\right)+\frac{5.33}{EI}=-\frac{11.08}{EI}$$



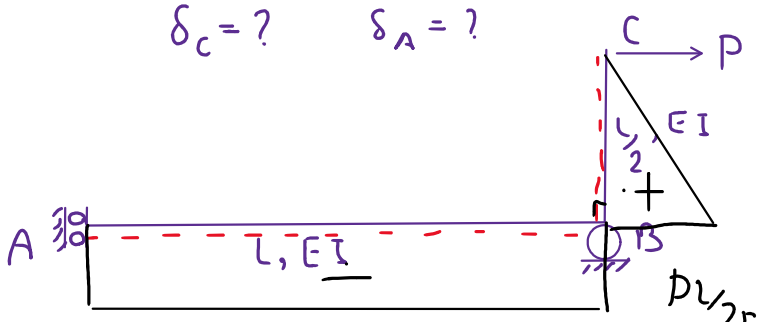
$$d\delta=\bar{x}\,d\theta=\bar{x}\frac{M}{EI}dx$$

Deflections4

Thursday, November 9, 2023 11:34



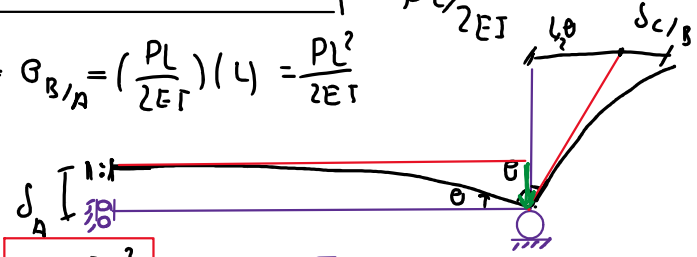
$\delta_C = ?$ $\delta_A = ?$



$$\delta_A = \delta_{B/A} = \left(\frac{PL}{2EI}\right)(L)\left(\frac{L}{2}\right) = \boxed{\frac{PL^3}{4EI}}$$

$$\theta_B = \theta_{B/A} = \left(\frac{PL}{2EI}\right)(L) = \frac{PL^2}{2EI}$$

$$\delta_C = L\theta_B + \delta_{C/B} = \frac{L}{2}\left(\frac{PL^2}{2EI}\right) + \frac{1}{2}\left(\frac{PL}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{2}{3}\times\frac{L}{2}\right) = \frac{6+1}{24} = \boxed{\frac{7}{24}\frac{PL^3}{EI}}$$



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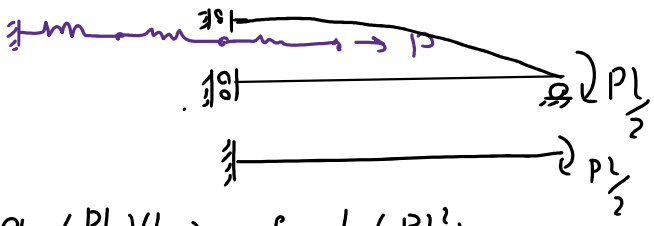
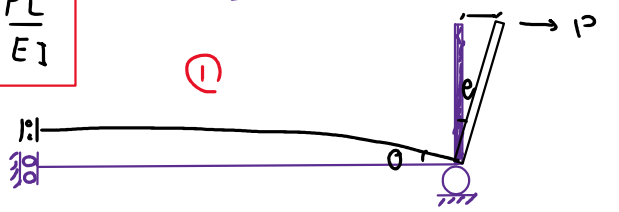


①



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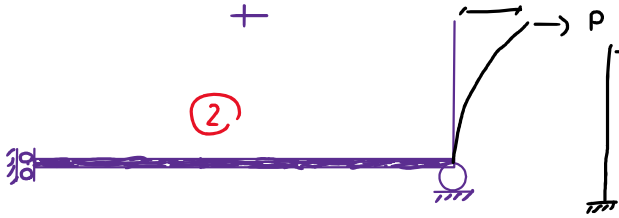


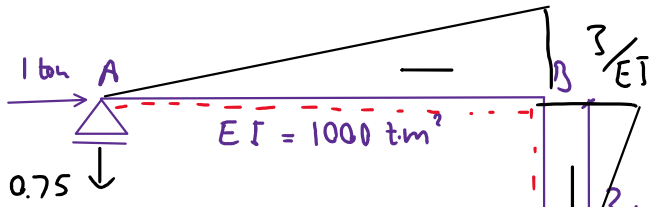
②

① $\theta = \left(\frac{PL}{2}\right)\left(\frac{L}{EI}\right)$, $\delta_C = \frac{L}{2}\left(\frac{PL^2}{2EI}\right)$

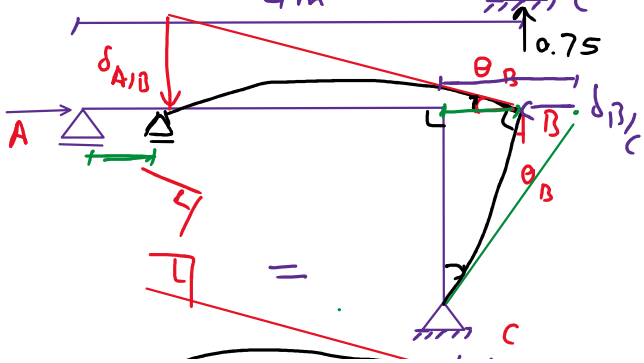
② $\delta_C = \frac{P\left(\frac{L}{2}\right)^3}{3EI}$

①+② $\delta_C = \frac{PL^3}{4EI} + \frac{PL^3}{24EI} = \frac{7}{24}\frac{PL^3}{EI}$

θ_C ، θ_B ، δ_A : أوجد



$$\delta_{A/B} = \frac{1}{2}\left(\frac{3}{EI}\right)(4)\left(\frac{2}{3}\times 4\right) = \frac{16}{EI}$$



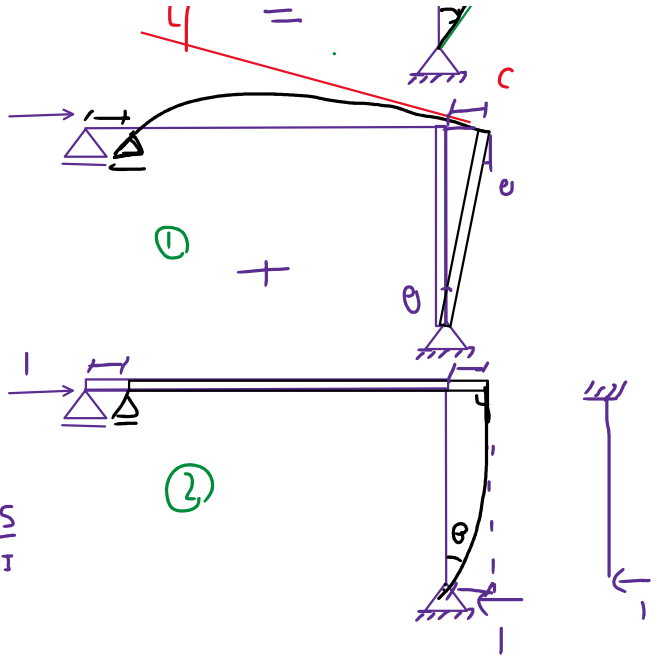
$$\theta_B = \frac{16/EI}{4} = \boxed{\frac{4}{EI}}$$

$$\theta_{C/B} = \theta_C - \theta_B = \frac{1}{2}\left(\frac{3}{EI}\right)(3) = \frac{4.5}{EI}$$

$$\theta_C - \left(\frac{-4}{EI}\right) = \frac{-4.5}{EI} \rightarrow \boxed{\theta_C = \frac{-8.5}{EI}}$$

$$\theta_C - \left(\frac{-4}{EI}\right) = \frac{-4.5}{EI} \rightarrow \boxed{\theta_C = \frac{-8.5}{EI}}$$

$$\delta_A = 3\left(\frac{8.5}{EI}\right) - \frac{1}{2}\left(\frac{3}{EI}\right)(3)(1) = \boxed{\frac{21}{EI}}$$



① $\theta_B = \frac{4}{EI}$ $\quad \delta_A = \frac{12}{EI}$ $\quad \theta_C = \frac{4}{EI}$

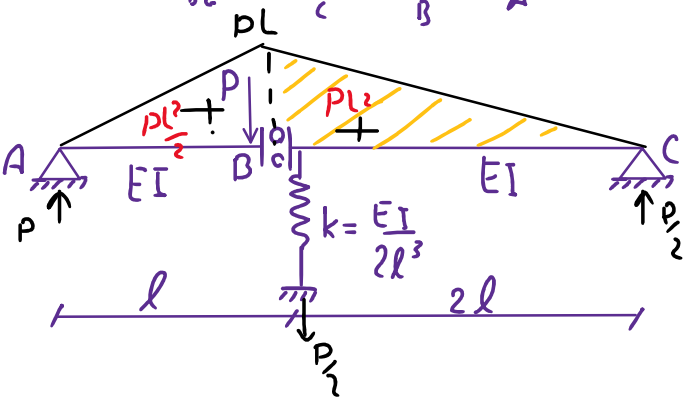
② $\theta_B = 0$ $\quad \delta_A = \frac{1 \times 3^3}{3EI} = \frac{9}{EI}$ $\quad \theta_C = \frac{1 \times 3^2}{2EI} = \frac{4.5}{EI}$

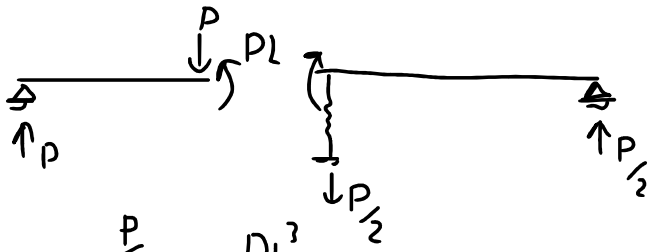
①+② $\theta_B = \frac{4}{EI}$ $\quad \delta_A = \frac{21}{EI}$ $\quad \theta_C = \frac{8.5}{EI}$

Deflection5

Thursday, November 9, 2023 13:42

مثال: δ_{BL} ، θ_C ، θ_B ، θ_A





$$\delta_{BR} = \frac{\frac{P}{2}}{\frac{EI}{2l^3}} = \frac{Pl^3}{EI}$$

$$\delta_{B/C} = \frac{1}{2}\left(\frac{Pl}{EI}\right)(2l)\left(\frac{2l}{3}\right) = \frac{2}{3}\frac{Pl^3}{EI}$$

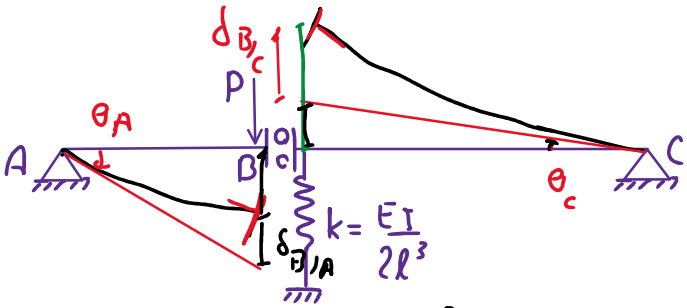
$$\theta_C = \frac{1}{2l}\left(\frac{Pl^3}{EI} - \frac{2}{3}\frac{Pl^3}{EI}\right) = \boxed{-\frac{Pl^2}{6EI}}$$

$$\theta_{C/B} = \theta_C - \theta_B = \frac{Pl^2}{EI} \rightarrow \boxed{\theta_B = \frac{-7}{6}\frac{Pl^2}{EI}}$$

$-Pl^2/6EI$

$$\theta_{B/A} = \theta_B - \theta_A = \frac{Pl^2}{2EI} \rightarrow \boxed{\theta_A = \frac{-5}{3}\frac{Pl^2}{EI}}$$

$-\frac{7}{6}\frac{Pl^2}{EI}$



$$\delta_{B/A} = \left(\frac{Pl^2}{2EI}\right)\left(\frac{l}{3}\right) = \frac{Pl^3}{6EI}$$

$$\delta_{BL} = l\left(-\frac{5}{3}\frac{Pl^2}{EI}\right) + \frac{Pl^3}{6EI} = \boxed{\frac{-3}{2}\frac{Pl^2}{EI}}$$

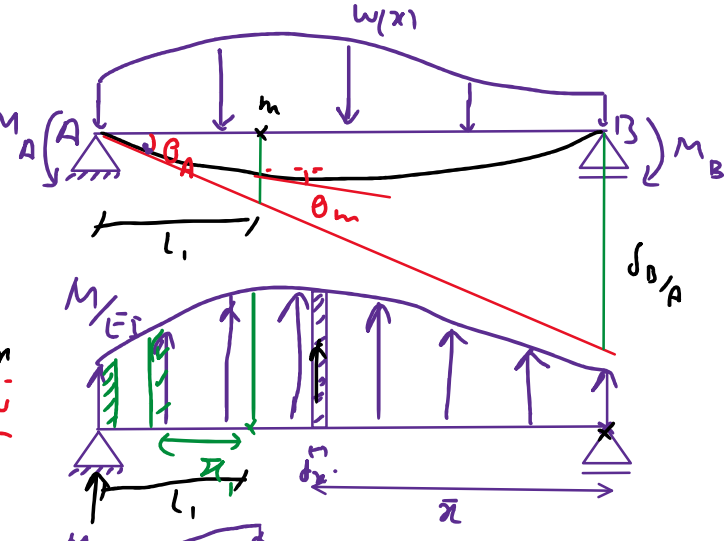
روش بار الاستیک

حالت خاصی از روش تیر مزدوج است (برای تیر دو سر مفصل)

لنگر - سطح

$$\theta_A = \frac{\delta_{B/A}}{L} = \frac{1}{L}\int_0^L \frac{M}{EI}\bar{x}\,dx$$

تیر اصلی



معادل تعادل تیر تانژ

$$\sum M_B = 0$$

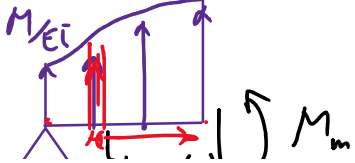
$$R_A(l) + \int_0^l \frac{M}{EI}\bar{x}\,dx \rightarrow R_A = \frac{1}{l}\int_0^l \frac{M}{EI}\bar{x}\,dx$$

تیر تانژ

$$\boxed{R_A = \theta_A}$$

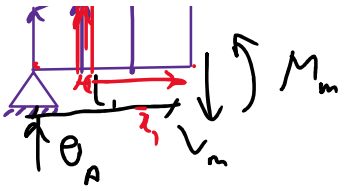
تیر تانژ — تیر اصلی

لنگر - سطح



لنگر-سطح

$$\theta_{m/A} = \theta_m - \theta_A = \int_0 \frac{M}{EI}\,dx \rightarrow \theta_m = \theta_A + \int_0^{L_1} \frac{M}{EI}\,dx$$



برش در تیر تا نوبه

$$\Sigma F_y = 0 \rightarrow \theta_A + \int_0^{L_1} \frac{M}{EI}\,dx - V_m = 0 \rightarrow V_m = \theta_A + \int_0^{L_1} \frac{M}{EI}\,dx$$

V_m = θ_m تیر اصلی
تیر تا نوبه

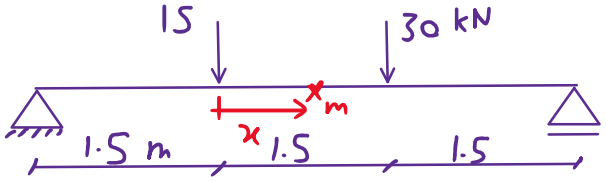
لنگر سطح

$$\delta_m = L_1\theta_A + \delta_{m/A} = L_1\theta_A + \int_0^{L_1} \frac{M}{EI}\bar{x}_1\,dx$$

لنگر در تیر تا نوبه

M_m = δ_m تیر اصلی
تیر تا نوبه

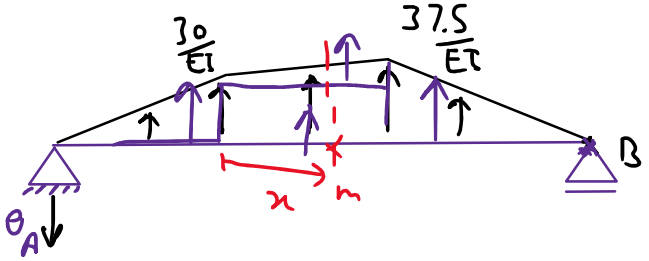
$\delta_{m\,ex}$ تیر اصلی = $M_{m\,ex}$ تیر تا نوبه



$+\circlearrowleft \Sigma M_B = 0$

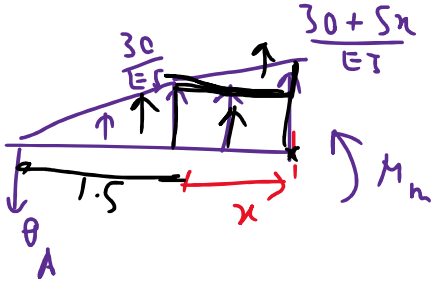
$$-4\theta_A + \frac{1}{2}\left(\frac{30}{EI}\right)(1.5)(3.5) + \left(\frac{30}{EI}\right)(1.5)(2.25)$$

$$+ \frac{1}{2}\left(\frac{7.5}{EI}\right)(1.5)(2) + \frac{1}{2}\left(\frac{37.5}{EI}\right)(1.5)(1) = 0$$



$$\theta_A = \frac{48.75}{EI}$$

$\Sigma F_y = 0$



$$-\frac{48.75}{EI} + \frac{1}{2}\left(\frac{30}{EI}\right)(1.5) + \frac{1}{2}\left(\frac{30}{EI} + \frac{30+5x}{EI}\right)x = 0 \rightarrow x = 0.819$$

$$M_m = -\frac{48.75}{EI} \times 2.319 + \frac{22.5}{EI} \times 1.319 + \frac{30}{EI}\frac{0.819^2}{2} + \frac{1}{2}\left(\frac{5 \times 0.819}{EI}\right) \times \frac{0.819^2}{3} = -\frac{72.85}{EI}$$

Deflection6

Monday, November 13, 2023 12:43

روش تیر مزدوج

بررسی معادلات دیفرانسیل تیرها

$$y'' = \frac{M}{EI}$$

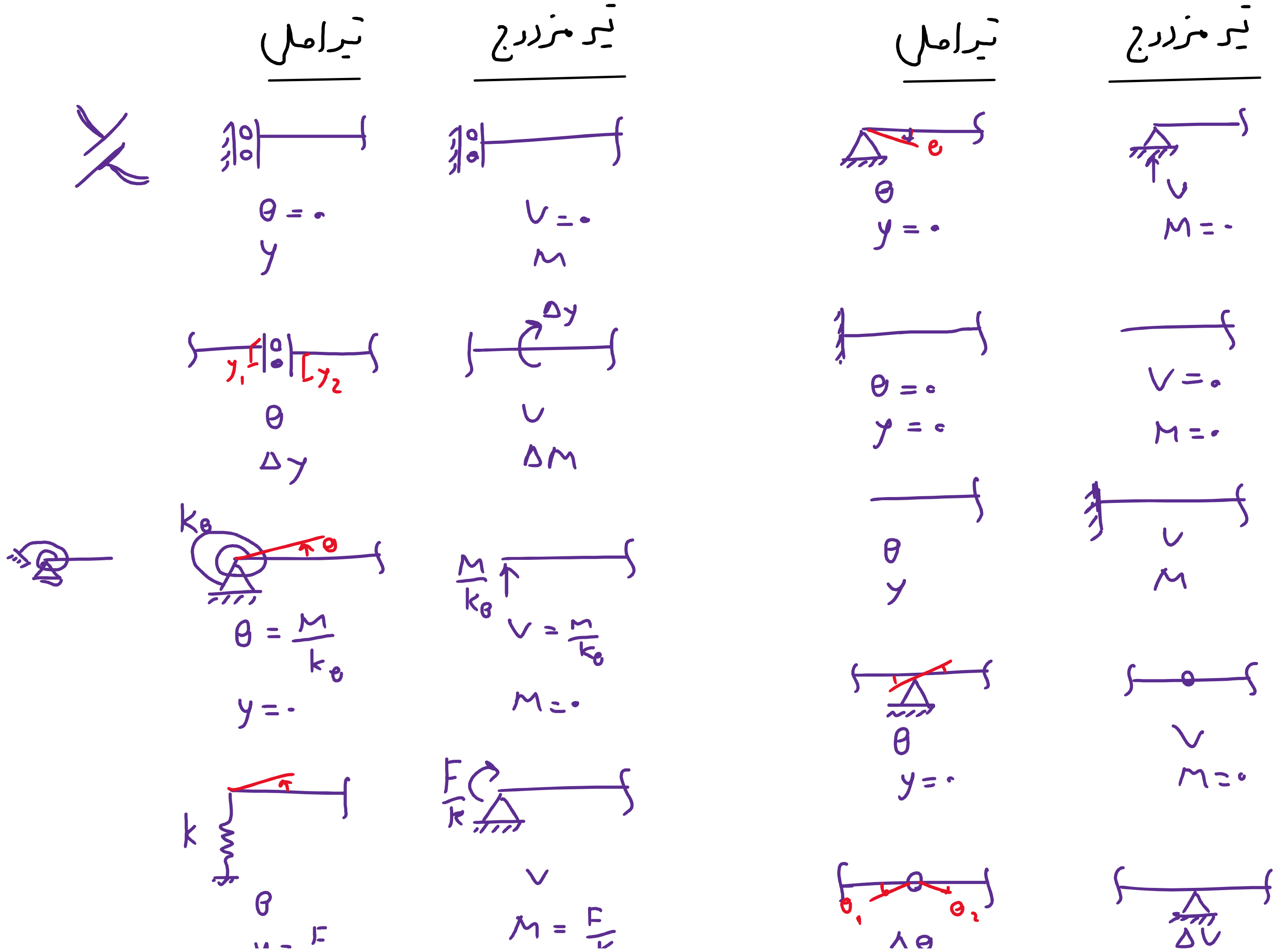
$$\begin{cases} \dfrac{dV}{dx} = w \\[2mm] \dfrac{dM}{dx} = V \end{cases} \qquad \begin{cases} \dfrac{d\theta}{dx} = \dfrac{M}{EI} \\[2mm] \dfrac{dy}{dx} = \theta \end{cases}$$

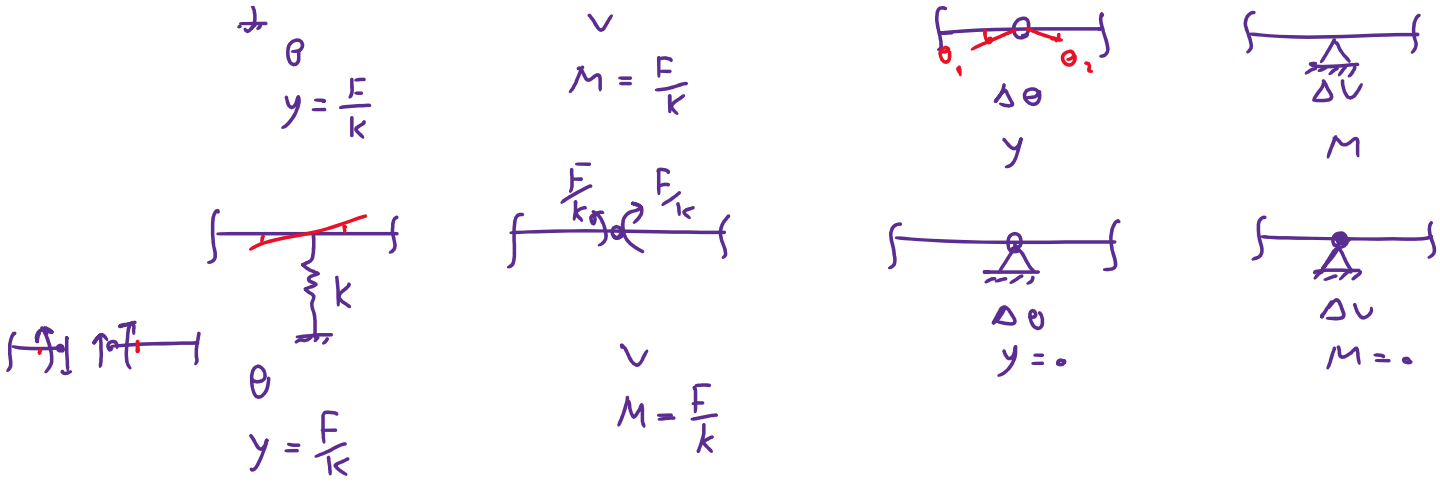
(V ← θ ، w ← M/EI ، M ← y)

* از آنجایی که معادلات دیفرانسیل فوق دقیقاً یکسان است، می‌توان فرض کرد که تیری نیز وجود دارد که تحت بار الاستیک $\frac{M}{EI}$ قرار می‌گیرد و در آن (تیر مزدوج)، برش (V) و لنگر (M) برابر است با چرخش (θ) و تغییر مکان (y) در تیر اصلی.

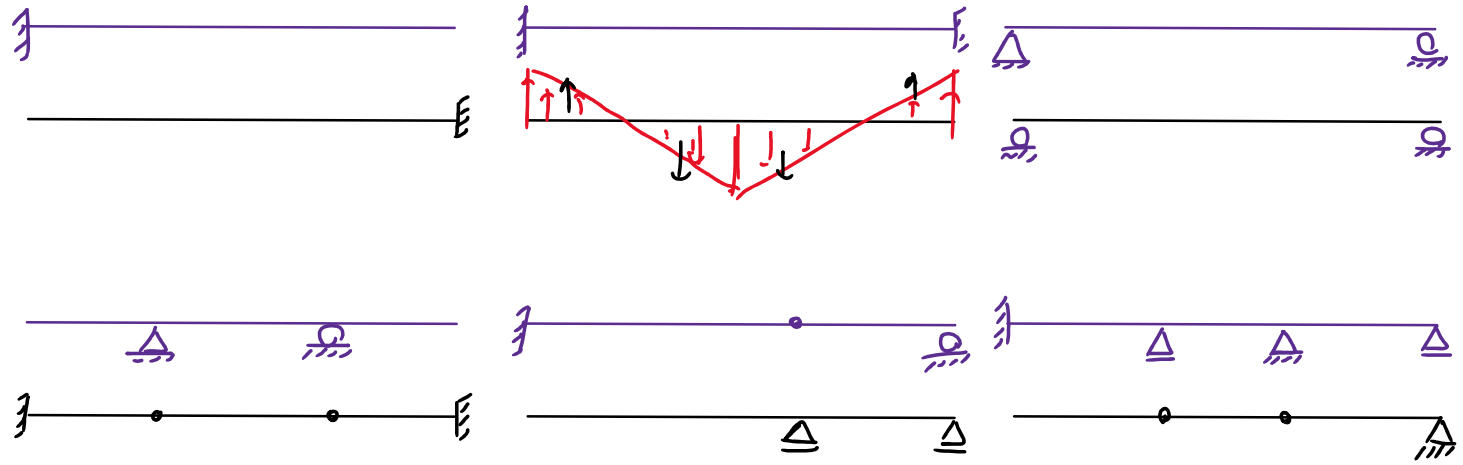
* برای تضمین صحت این شبیه‌سازی، علاوه بر معادلات دیفرانسیل، شرایط مرزی نیز باید ارضا شود.

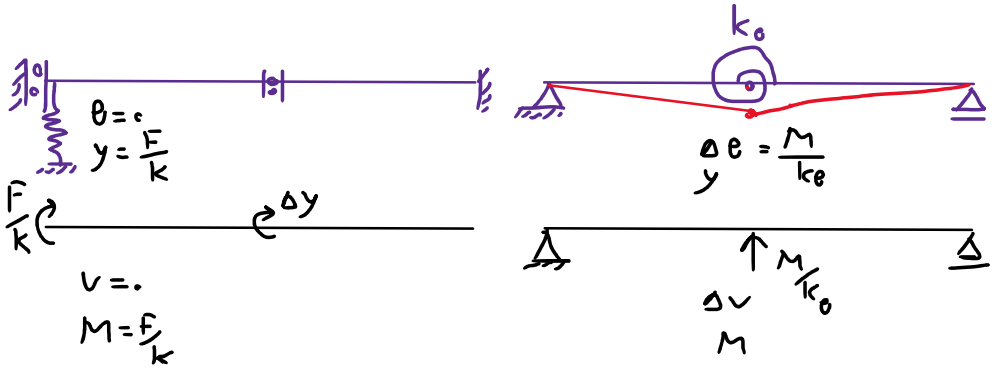
بنابراین داریم:





مثال: تیر مزدوج معادل تیرهای شکل زیر را رسم نمایید.

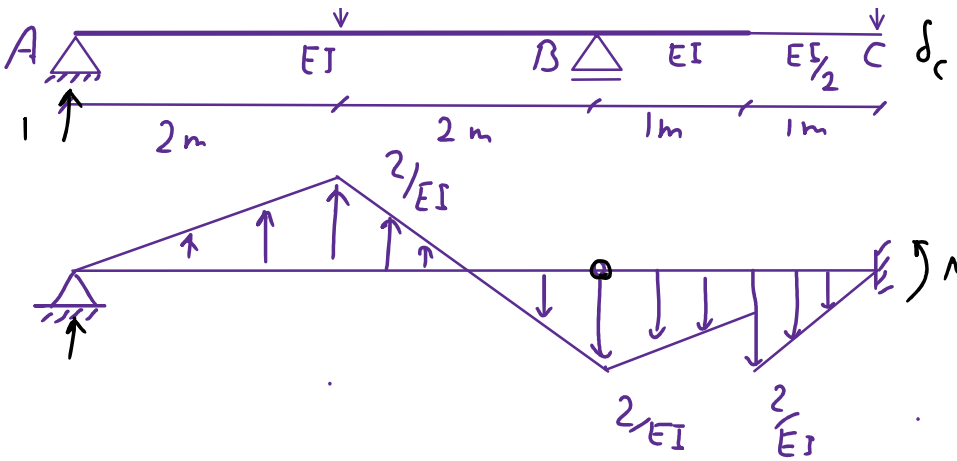




* چنانکه مشاهده می‌شود، بعضی از تیرهای مزدوج ناپایدار هستند ولی تحت بارگذاری خاص $\frac{M}{EI}$ پایدار می‌باشند.

مثال: $\delta_c = ?$

$$\delta_c = \frac{-7}{3EI}$$



$$\curvearrowleft \Sigma M_A = 0 \rightarrow \frac{1}{2}\left(\frac{2}{EI}\right)(2)\left(\frac{2}{3}\times 2\right)+\frac{1}{2}\left(\frac{2}{EI}\right)(1)\left(\frac{7}{3}\right)-\frac{1}{2}\left(\frac{2}{EI}\right)(1)\left(\frac{11}{3}\right)-V_B(4)=$$

θ_B = نیراصل $= V_B = \frac{1}{3EI}$

$$\curvearrowright \Sigma M_c = 0 \rightarrow M_c = \left(\frac{1}{3EI}\right)(2)-\frac{1}{2}\left(\frac{2}{EI}\right)(2)\left(\frac{2}{3}\times 2\right)-\frac{1}{2}\left(\frac{1}{EI}\right)(1)\left(\frac{2}{3}\times 1\right)=\frac{-7}{3EI}$$

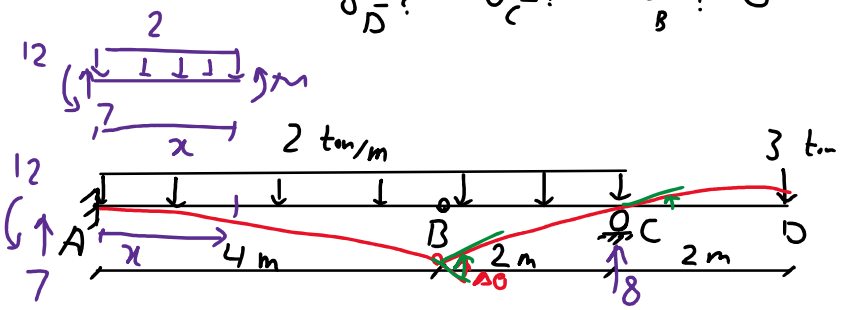
δ_c = نیراکل $= M_c = \frac{-7}{3EI}$

$$\delta_c = 2\theta_B + \delta_{c/B} = 2\left(\frac{1}{3EI}\right)+\frac{1}{2}\left(\frac{-2}{EI}\right)(2)\left(\frac{4}{3}\right)+\frac{1}{2}\left(\frac{-1}{EI}\right)(1)\left(\frac{2}{3}\right)$$

Deflection7

Thursday, November 16, 2023 18:40

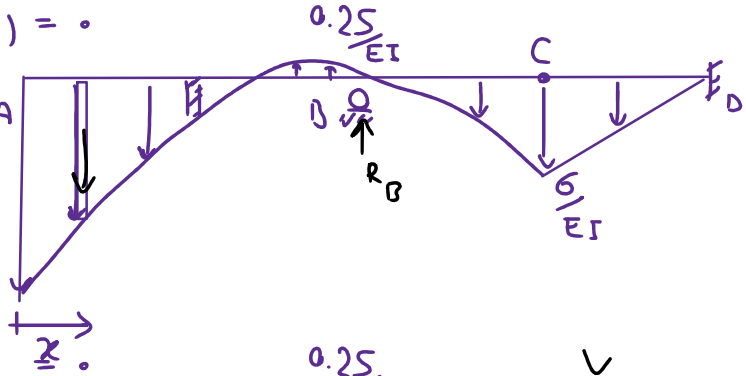
مثال: $\delta_D = ?$ $\theta_C = ?$ $\Delta\theta_B = ?$



$M = -12 + 7x - x^2$

$\circlearrowleft \sum M_C = 0 \rightarrow$

$$\int_0^6 \frac{1}{EI}(-12+7x-x^2)(6-x)\,dx + R_B(2) = 0$$

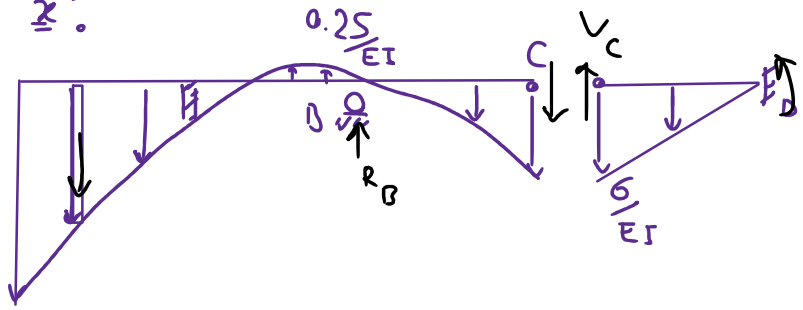


$$\frac{-72}{EI} + R_B(2) = 0 \rightarrow \Delta\theta = R_B = \frac{36}{EI}$$

شیب تیر آزاد

$\sum F_y = 0$

$$\int_0^6 \frac{1}{EI}(-12+7x-x^2)\,dx + \frac{36}{EI} - V_C = 0$$



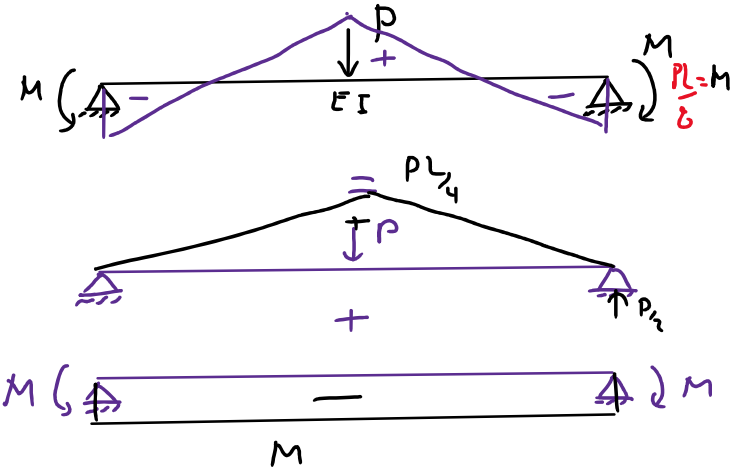
$$\frac{-18}{EI} + \frac{36}{EI} - V_C = 0 \rightarrow \theta = V_C = \frac{18}{EI}$$

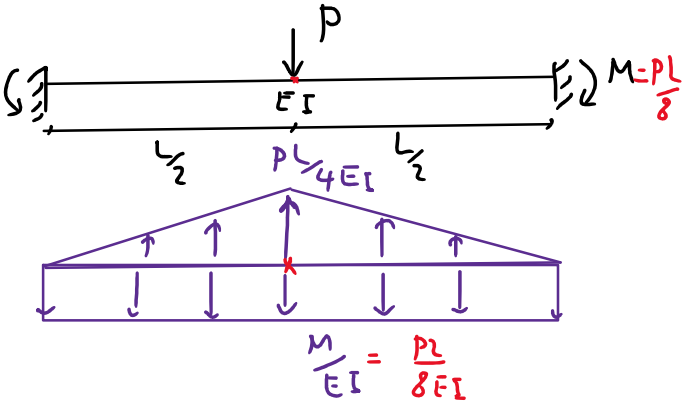
شیب تیر آزاد

$$M_D = \left(\frac{18}{EI}\right)(2) - \frac{1}{2}\left(\frac{6}{EI}\right)(2)\left(\frac{2}{3}\times 2\right) = \frac{28}{EI}$$

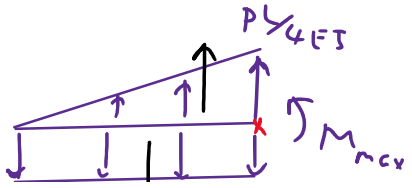
$$\delta_D = M_D = \frac{28}{EI} \uparrow$$

مثال: تغییرمکان ماکزیمم تیر شکل زیر را به دست آورید.

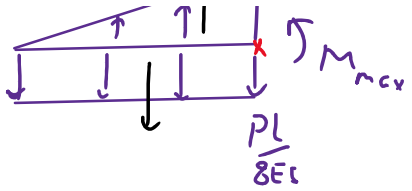




$$\sum F_y = 0 \rightarrow \frac{1}{2}\left(\frac{PL}{4EI}\right)(L) - \left(\frac{M}{EI}\right)(L) = 0 \rightarrow M = \frac{PL}{8}$$



$$M = \frac{1}{2}\left(\frac{PL}{4EI}\right)(L)\left(\frac{1}{3}\times L\right) - \left(\frac{PL}{4EI}\right)(L)(L) =$$



$$M_{max} = \frac{1}{2}\left(\frac{PL}{4EI}\right)\left(\frac{L}{2}\right)\left(\frac{1}{3}\times\frac{L}{2}\right) - \left(\frac{PL}{8EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{4}\right) =$$

$$\frac{2-3}{32\times3\times2}\,\frac{PL^3}{EI}$$

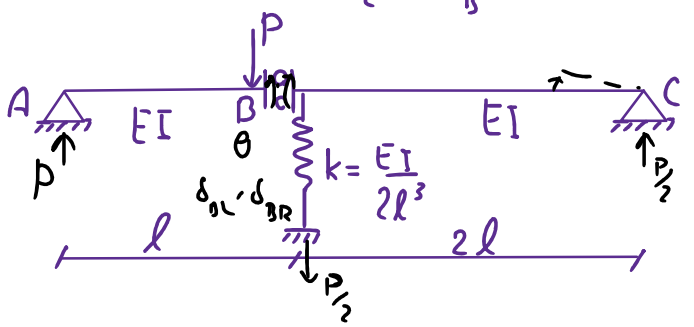
$$\delta_{max} = M_{max} = \frac{-1}{192}\,\frac{PL^3}{EI}$$

Deflection8

Friday, November 17, 2023 2:05

$$\delta_{BR} = \frac{\frac{P}{2}}{\frac{EI}{2l^3}} = \frac{Pl^3}{EI}$$

مثال: δ_{BL} ، θ_C ، θ_B ، θ_A



$$\sum M_B = 0$$

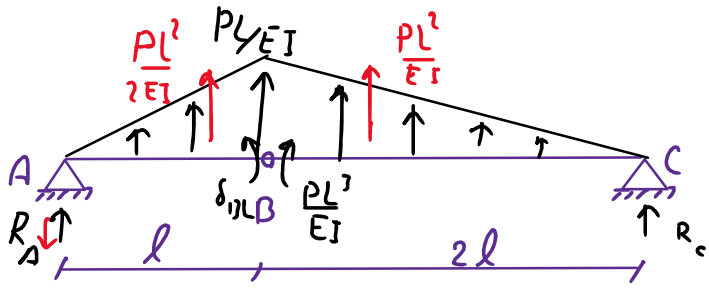
$$R_C(2l) + \left(\frac{Pl^2}{EI}\right)\left(\frac{1}{3}\times 2l\right) - \frac{Pl^3}{EI} = 0$$

$$\theta_C = v_C = \frac{-1}{6}\frac{Pl^2}{EI}$$

$$\sum F_y = 0$$

$$R_A + \frac{Pl^2}{2EI} + \frac{Pl^2}{EI} + \frac{1}{6}\frac{Pl^2}{EI} = 0$$

$$\theta_A = v_A = \frac{-10}{6}\frac{Pl^2}{EI}$$

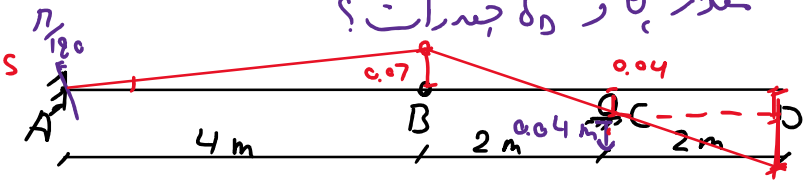


$$v_B = \frac{-10}{6}\frac{Pl^2}{EI} + \frac{Pl^2}{2EI} \rightarrow \theta_B = v_B = \frac{-7}{6}\frac{Pl^2}{EI}$$

تیر مزدوج تیر اصلی

$$\delta_{BL} = \left(\frac{-10}{6}\frac{Pl^2}{EI}\right)(l) + \left(\frac{Pl^2}{2EI}\right)\left(\frac{l}{3}\right) \rightarrow \delta_{BL} = \frac{-3}{2}\frac{Pl^2}{EI}$$

مثال: اگر در تیر شکل زیر تکیه‌گاه C به اندازه‌ی 0.04 نشست و تکیه‌گاه A به اندازه‌ی $\frac{\pi}{180}$ دوران پادساعتگرد داشته باشد، مقدار θ_C و δ_D چقدر است؟



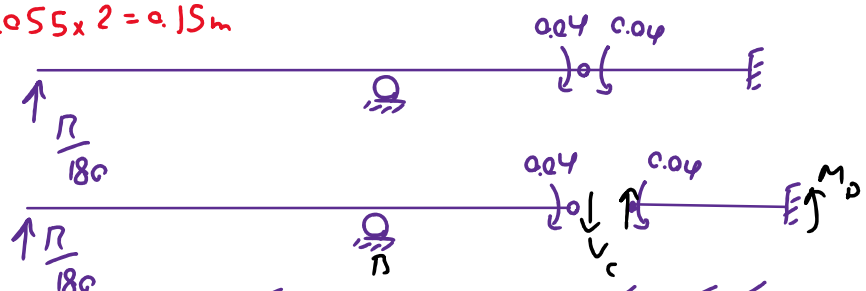
$$\sum M_B = 0$$

$$\theta_C = \frac{0.07 + 0.04}{2} = 0.055$$

$$\left(\frac{\pi}{180}\right)(4) + 0.04 + v_C(2) = 0$$

$$\theta_C = v_C = -0.055 \text{ rad}$$

$$\delta_D = 0.04 + 0.055 \times 2 = 0.15 \text{ m}$$



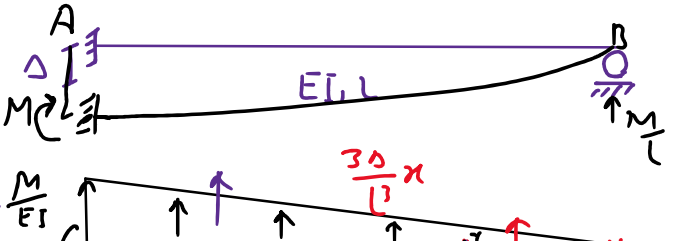
$$M_D = (-0.055)(2) - 0.04 = -0.15 \text{ m}$$

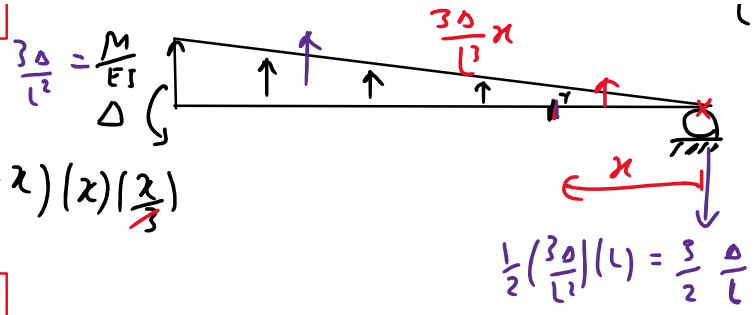
$$\delta_D = M_D = -0.15 \text{ m}$$

مثال: اگر تکیه‌گاه گیردار A به اندازه‌ی Δ نشست کند؛ لنگر انتهای گیردار و تابع تغییر شکل تیر را بدست آورید.

$$\sum M_A = 0$$

$$\frac{1}{2}\left(\frac{M}{EI}\right)(L)\left(\frac{2}{3}L\right) - \Delta = 0 \rightarrow M = \frac{3EI}{L^2}\Delta$$





$$\delta(x) = M(x) = -\frac{3}{2}\left(\frac{\Delta}{l}\right)x + \frac{1}{2}\left(\frac{3\Delta}{l^3}x\right)(x)\left(\frac{x}{3}\right)$$

تیر مرجع تیر اصلی

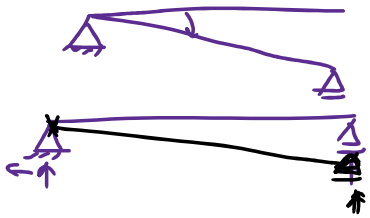
$$\boxed{\delta(x) = \left(-\frac{3}{2}\frac{\Delta}{l}\right)x + \left(\frac{1}{2}\frac{\Delta}{l^3}\right)x^3}$$

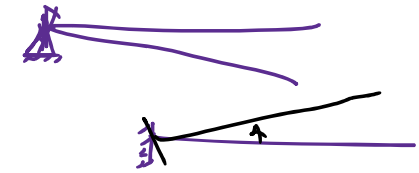
* نشست تکیه گاهی و تغییر دما

سازه نامعین ← نیرو ایجاد می کند

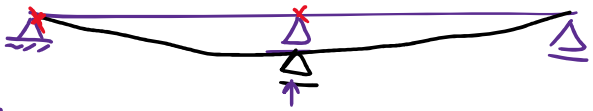
سازه نامعین ← نیرو ایجاد می کند

۳ معادله ۳ مجهول





$$[A]\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$



۳ معادله تعادل
۴ مجهول تکیه گاهی

$$[A]\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{cases} a_1R_1 + c_1R_2 + c_3R_3 + c_4R_4 = 0 \\ b_1R_1 + \cdots = 0 \\ c_1R_1 + \cdots \quad c_4R_4 = 0 \end{cases}$$

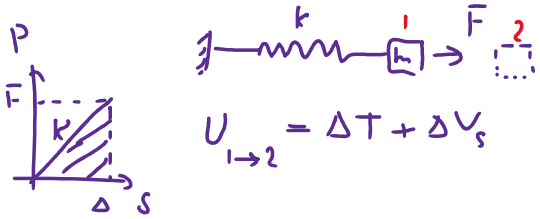
Deflection Energy1

Thursday, November 23, 2023 8:26

محاسبه تغییر شکل ها به روش های انرژی

روش های انرژی برای محاسبه تغییر شکل ها عبارتند از:

- روش کار حقیقی
- روش بار واحد (روش کار مجازی)
- روش کاستیلیانو
- قانون بتی و قانون ماکسول



$U_{1\to 2} = \Delta T + \Delta V_s$

$\frac{1}{2} F_s \Delta = \frac{1}{2} k \Delta^2$

روش کار حقیقی

بنابراصل بقای انرژی، کار انجام شده توسط بارهای خارجی برابراست با کار انجام شده توسط نیروهای داخلی (انرژی کرنش ذخیره شده در اعضای سازه)

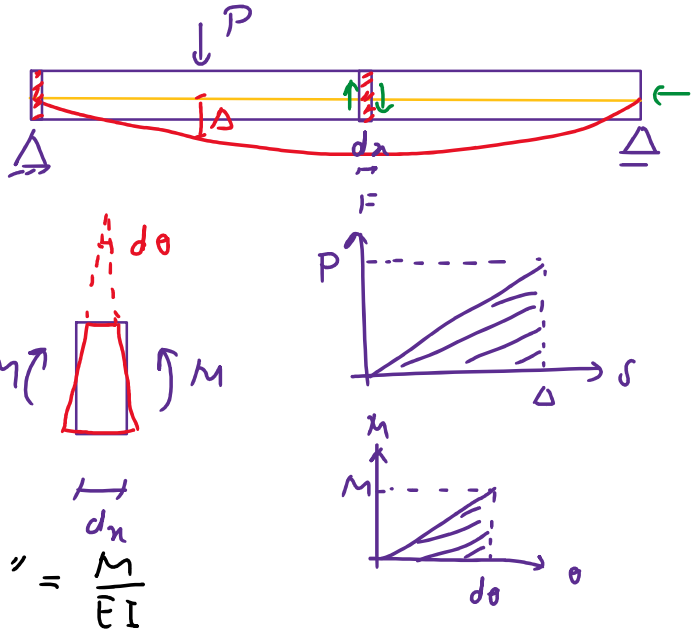
$$W_{ext} = (W_{int} = U)$$

به طور مثال برای یک تیر کار خارجی و انرژی کرنشی ذخیره شده را می توان به صورت زیر نوشت:

$$W_{ext} = \frac{1}{2} P \Delta$$

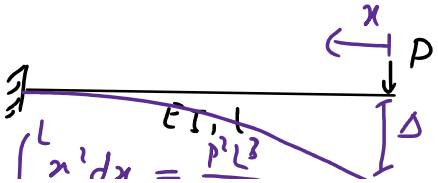
$$dW_{int} = \frac{1}{2} M d\theta = \frac{1}{2} M \left(\frac{M}{EI} dx\right)$$

$$W_{int} = \frac{1}{2} \int_0^L \frac{M^2}{EI} dx$$



$$\frac{d\theta}{dx} = y'' = \frac{M}{EI}$$

مثال: تغییرمکان قائم سر آزاد تیر زیر را به دست آورید.



$$\begin{cases} W_{ext} = \frac{1}{2} P \Delta \\ W_{int} = U = \frac{1}{2} \int_0^L \frac{M^2}{EI} dx = \frac{1}{2} \int_0^L \frac{(-Px)^2}{EI} dx = \frac{P^2}{2EI} \int_0^L x^2 dx = \frac{P^2 L^3}{6EI} \end{cases}$$

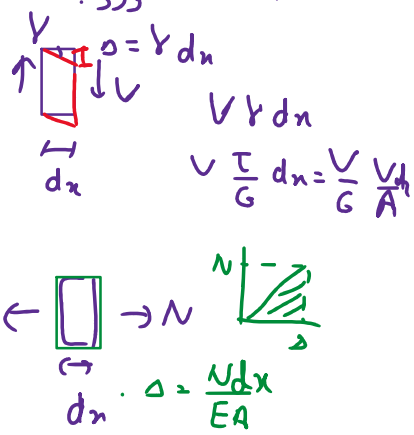
$$W_{ext} = \frac{1}{2}P\Delta$$

$$W_{int} = U = \frac{1}{2}\int_0^L \frac{M^2}{EI}dx = \frac{1}{2}\int_0^L \frac{(-Px)^2}{EI}dx = \frac{P^2}{2EI}\int_0^L x^2 dx = \frac{P^2L^3}{6EI}$$

$$\frac{1}{2}P\Delta = \frac{P^2L^3}{6EI} \rightarrow \boxed{\Delta = \frac{PL^3}{3EI}}$$

* انرژی کرنشی ناشی از سایر نیروها (نیروی محوری، نیروی برشی و لنگر پیچشی) را نیز می‌توان به روش بالا به دست آورد.

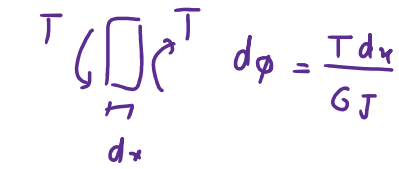
$$U = U_{\text{خمش}} + U_{\text{نیروی محوری}} + U_{\text{برش}} + U_{\text{پیچش}}$$



$$\boxed{U = \frac{1}{2}\int_0^L \frac{M^2dx}{EI} + \frac{1}{2}\int_0^L \frac{N^2dx}{EA} + \frac{1}{2}k\int_0^L \frac{V^2dx}{GA} + \frac{1}{2}\int_0^L \frac{T^2dx}{JG}}$$

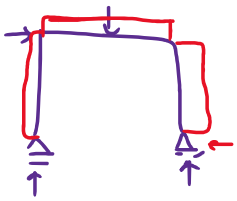
$$\tau = \frac{VQ}{It}$$ ▭ ▷ I ▭ $$\tau \cong \frac{V}{A}$$

* k در مقاطع مستطیلی برابر 1.2، در مقاطع دایره‌ای برابر $\frac{10}{9}$ و در مقاطع I شکل برابر 1 است.



$$\boxed{U = \sum_i \frac{1}{2}\frac{N_i^2 L_i}{E_i A_i}}$$ خرپاها

$$\boxed{U = \frac{1}{2}\int_0^L \frac{M^2}{EI}dx}$$ تیرها و قاب‌ها



* دامنه کاربرد روش کار مجازی، بسیار محدود است و در تحلیل سازه به ندرت استفاده می‌شود. چرا که این روش فقط تغییر مکان در یک بار متمرکز را می‌دهد.

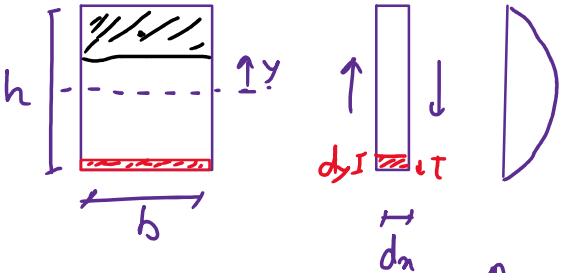
انرژی کرنشی ناشی از برش

$$Q = b\left(\frac{h}{2}-y\right)\frac{1}{2}\left(\frac{h}{2}+y\right)$$

$$Q = \frac{b}{8}(h^2-4y^2)$$

$$U = \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{T^2}{G} b\,dy\,dx$$

$$U = \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{VQ}{Ib}\right)^2 \frac{b}{G}\,dy\,dx = \frac{V^2dx}{I^2bG}\int Q^2 dy$$



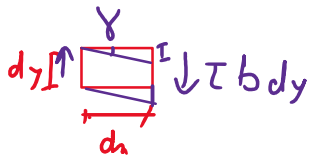
$$\int_{-\frac{h}{2}}^{\frac{h}{2}} Q^2 dy = \int \left(\frac{b}{8}\right)^2 (h^2 - 4y^2)^2 dy =$$

$$\frac{b^2}{64} \int (h^4 + 16y^4 - 8h^2y^2)\, dy =$$

$$\frac{b^2}{64}\left[h^4 y + \frac{16}{5}y^5 - \frac{8h^2}{3}y^3\right]_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{b^2}{64} \times 2\left[h^4\left(\frac{h}{2}\right) + \frac{1}{5}\left(\frac{h}{2}\right)^5 - \frac{8h^2}{3}\left(\frac{h}{2}\right)^3\right] \quad \frac{15+3-10}{30} h^5$$

$$\frac{b^2}{8} \times 2 \times \frac{8}{15} h^5 = \frac{1}{120} b^2 h^5$$

$$\boxed{\int Q^2 dy = \frac{1}{120} b^2 h^5}$$

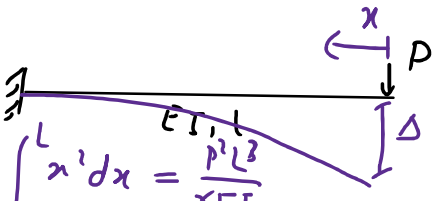


$$dU = (\tau b\, dy)(\gamma dx) = \frac{\tau^2}{G} b\, dy\, dx$$

$$U = \frac{V^2 dx}{I^2 b G} \int Q^2 dy = \frac{V^2 dx}{I^2 b G} \times \frac{1}{120} b^2 h^5 = \frac{V^2 dx}{\left(\frac{1}{12} b h^3\right) G} \times \frac{1}{10} h^2 = 1.2 \frac{V^2 dx}{GA}$$

$$\boxed{U = \frac{1}{2} \int_0^L \frac{1.2 V^2}{GA} dx}$$

مثال را با در نظر گرفتن تغییر شکل برشی حل کنید.



$$\begin{cases} W_{ext} = \frac{1}{2} P \Delta \\ W_{int} = U = \frac{1}{2} \int_0^L \frac{M^2}{EI} dx = \frac{1}{2} \int_0^L \frac{(-Px)^2}{EI} dx = \frac{P^2}{2EI} \int_0^L x^2 dx = \frac{P^2 L^3}{6EI} \end{cases}$$

$$\frac{1}{2} P \Delta = \frac{P^2 L^3}{6EI} \rightarrow \boxed{\Delta = \frac{PL^3}{3EI}}$$

$$U = U_{\text{خمشی}} + U_{\text{برشی}}$$

$$U_{\text{برشی}} = \frac{1}{2} \int_0^L \frac{1.2 V^2}{GA} dx = \frac{6}{10} \frac{1}{GA} \int_0^L P^2 dx = \frac{6}{10} \frac{P^2 L}{GA}$$

$$G = \frac{E}{2(1+\nu)} = \frac{E}{2(1+0.2)}$$

$$\frac{1}{2} P \Delta = \frac{P^2 L^3}{6EI} + \frac{6}{10} \frac{P^2 L}{GA} \rightarrow \Delta = \frac{PL^3}{3EI} + \frac{6}{10} \frac{PL}{GA} = \frac{PL^3}{3EI}\left(1 + \frac{18}{10} \frac{EI}{GA}\right) = \frac{PL^3}{3EI}\left(1 + 2.6 \frac{\frac{1}{12}h^2}{A}\right)$$

$$\frac{1}{2} P \Delta = \frac{P L^3}{6EI} + \frac{6}{10} \frac{P L}{GA} \rightarrow \Delta = \frac{P L^3}{3EI} + \frac{6}{5} \frac{PL}{GA} = \frac{PL^3}{3EI} \left(1 + \frac{18}{5} \frac{EI}{GA} \times \frac{1}{L^2} \right)$$

$2.6 \quad \frac{1}{12} h^2$

$$\boxed{\Delta = \frac{PL^3}{3EI} \left(1 + 0.78 \frac{h^2}{L^2} \right)}$$

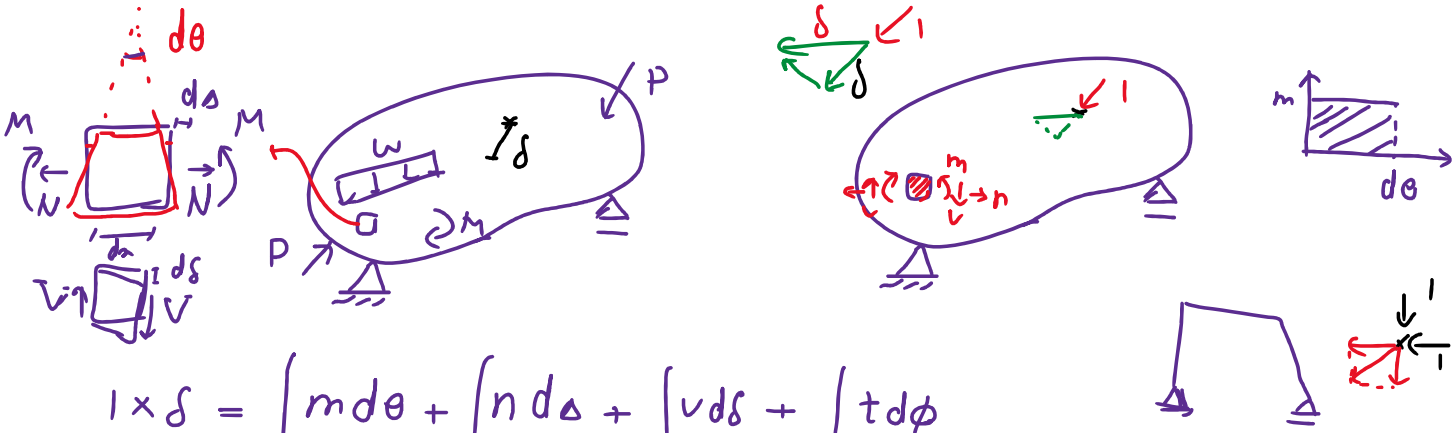
Deflection Energy2

Thursday, November 23, 2023 10:08

روش بار واحد

این روش عمومی ترین روش محاسبه تغییر شکل ها در سازه است که تقریبا بر کلیه محدودیت های روش های دیگر فائق می آید و زیر مجموعه روش های کار مجازی می باشد.

در این روش ابتدا یک بار مجازی واحد در محل و جهت تغییر شکل خواسته شده بر روی سازه قرار می گیرد. سپس کلیه بارهای خارجی، نشست تکیه گاهی، تغییر دما و نقص عضو به سازه اعمال می شود. در این صورت کار انجام شده توسط بار واحد برابر است با کار انجام شده توسط نیروهای مجازی داخلی ناشی از بار واحد.



$$1 \times \delta = \int m\,d\theta + \int n\,d\Delta + \int v\,d\delta + \int t\,d\phi$$

$$1 \times \delta = \int \frac{mM}{EI}\,dx + \int \frac{nN}{EA}\,dx + k\int \frac{vV}{GA}\,dx + \int \frac{tT}{GJ}\,dx$$

m, n, v, t : کنش خمشی، نیروی محوری، نیروی برشی و کنش پیچشی ناشی از بار واحد

M, N, V, T : - - - ناشی از بارگذاری خارجی

* کار داخلی انجام شده توسط بار واحد با انتگرال گیری بر روی کلیه اعضای سازه به دست می آید.

خرپاها

$$1 \times \delta = \sum_i \frac{n_i N_i L_i}{E_i A_i}$$

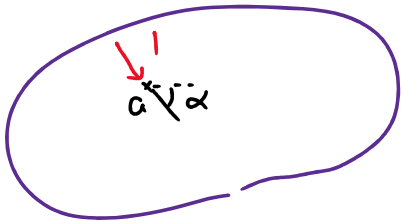
تیرها و قاب ها

$$1 \times \delta = \int \frac{mM}{EI}\,dx$$

روش اعمال بار واحد

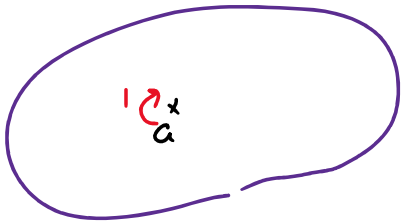
روش اعمال بار واحد

① تغییر مکان نقطه a در امتداد α



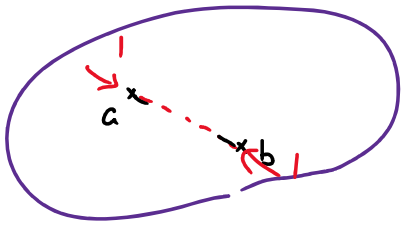
$$1 \times \delta = U$$

② چرخش نقطه a



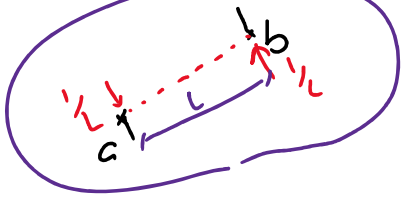
$$1 \times \theta = U$$

③ نزدیک و دور شدن دو نقطه a و b

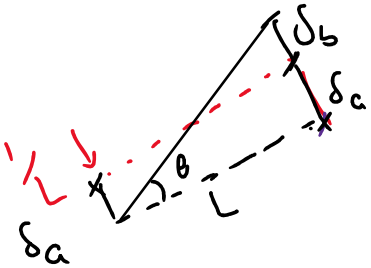


$$1 \times \delta_a + 1 \times \delta_b = U$$

④ دوران عضو ab یا چرخش در نقطه a و b نسبت به هم



$$\frac{1}{L}\delta_a + \frac{1}{L}\delta_b = U$$



$$\theta = \frac{1}{L}(\delta_a + \delta_b)$$

کار مجازی

بزرگ مجازی

$$1 \times \delta = \sum u\,dL$$

تغییر مکان حقیقی

$$\frac{1}{2}(1)(\delta) + \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + 1\cdot\Delta = \frac{1}{2}\sum u\cdot dL_1 + \frac{1}{2}\sum S\cdot dL + \sum u\cdot dL$$

بزرگ مجازی

$$1 \times \delta = \int \frac{mM}{EI}\,dx\,d\theta$$

تغییر مکان واقعی

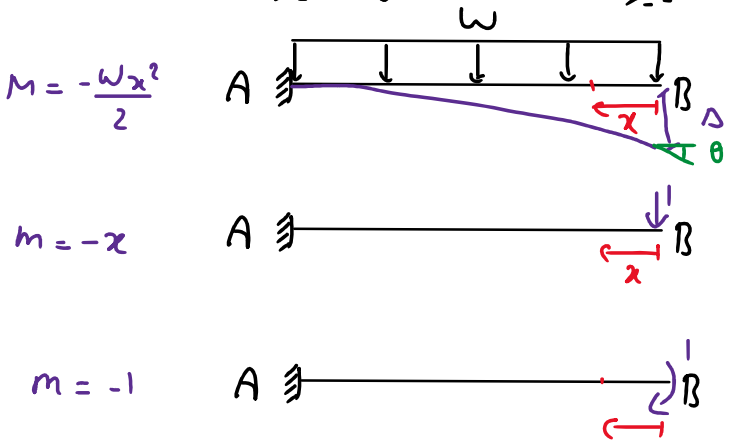
Deflection Energy3

Thursday, November 23, 2023 11:32

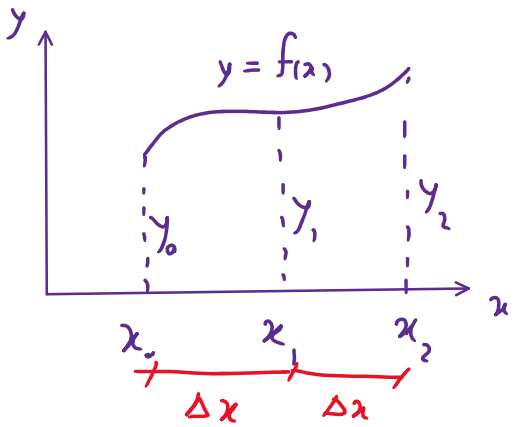
مثال: تغییر مکان و شیب انتهای آزاد تیر را به دست آورید.

$$1 \times \Delta = \int_0^L \frac{m M}{EI} dx$$

$$1 \times \Delta = \frac{1}{EI}\int_0^L (-x)\left(-\frac{wx^2}{2}\right) dx = \frac{w}{2EI}\frac{x^4}{4}\Big|_0^L = \boxed{\frac{wL^4}{8EI}}$$



$$1 \times \theta = \int_0^L \frac{m M}{EI} dx = \frac{1}{EI}\int_0^L (-1)\left(-\frac{wx^2}{2}\right) dx = \frac{w}{2EI}\int_0^L x^2 dx = \boxed{\frac{wL^3}{6EI}}$$



محاسبه عددی انتگرال $\int m M dx$

① روش سیمپسون

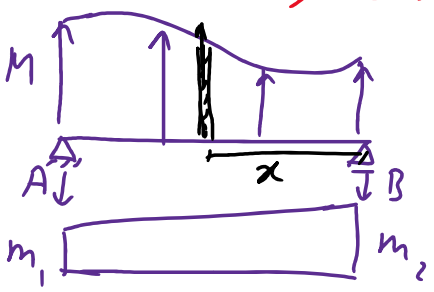
$$\int_{x_0}^{x_2} f(x)dx = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$$

سیمپسون $\frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + y_n)$

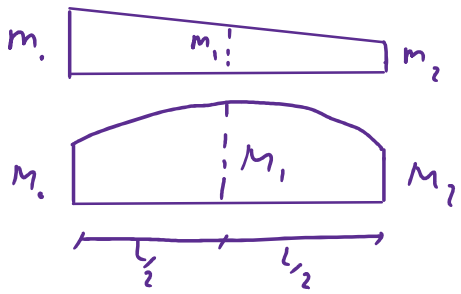
* اگر درجه عبارت زیر انتگرال ۳ یا کمتر باشد، جواب روش سیمپسون دقیق است.

② روش مور

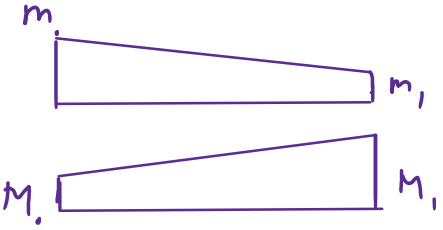
$$\int mM dx = m_1 A + m_2 B$$



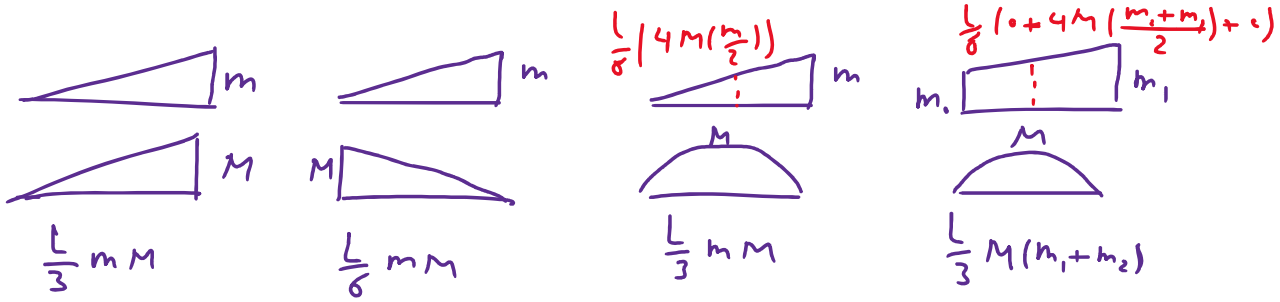
$$\int mM\,dx = \frac{L}{6}\left[m_0M_0 + 4m_1M_1 + m_2M_2\right]$$



$$\int mM\,dx = \frac{L}{6}\left[m_0M_0 + 4\left(\frac{m_0+m_1}{2}\right)\left(\frac{M_0+M_1}{2}\right) + m_1M_1\right]$$



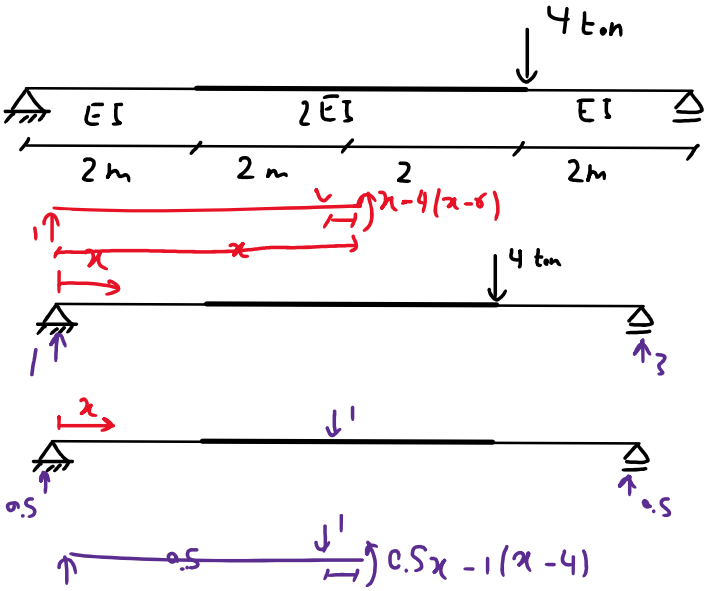
$$\int mM\,dx = \frac{L}{6}\left[2m_0M_0 + m_0M_1 + m_1M_0 + 2m_1M_1\right]$$



مثال: تغییر مکان قائم وسط دهانه تیر را به دست آورید ($EI = 600\,t.m^2$)

$$1\times\delta = \int_0^L \frac{mM}{EI}\,dx$$

$$t\times m = \frac{(t.m)^2}{t.m^2}\,m$$



$$M = \begin{cases} x & x \le 6 \\ -3x+24 & 6 < x < 8 \end{cases}$$

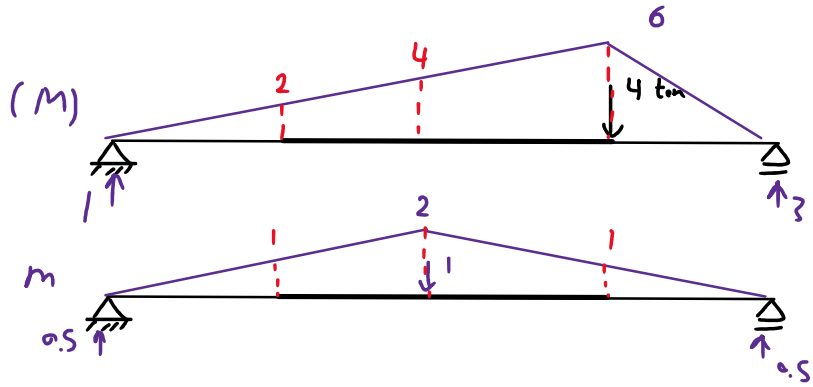
$$m = \begin{cases} \frac{1}{2}x & x \le 4 \\ -\frac{1}{2}x+4 & 4 \le x \le 8 \end{cases}$$

$$1\times\delta = \frac{1}{EI}\int_0^2 \left(\frac{x}{2}\right)(x)\,dx + \frac{1}{2EI}\int_2^4 \left(\frac{x}{2}\right)(x)\,dx + \frac{1}{2EI}\int_4^6 \left(-\frac{x}{2}+4\right)(x)\,dx + \frac{1}{EI}\int_6^8 \left(-\frac{x}{2}+4\right)(-3x+24)\,dx$$

$$1\times\delta = \frac{1}{\quad}\left[\frac{x^3}{\quad}\Big|_0^2 + \frac{1}{\quad}\frac{x^3}{\quad}\Big|_2^4 + \frac{1}{\quad}\left(-\frac{x^3}{\quad}+2x^2\right)\Big|_4^6 + \left(\frac{x^3}{\quad}-12x^2+96x\right)\Big|_6^8\right]$$

$$1 \times \delta = \frac{1}{EI}\left[\frac{x^3}{6}\Big|_0^2 + \frac{1}{2}\frac{x^3}{6}\Big|_2^4 + \frac{1}{2}\left(-\frac{x^3}{6} + 2x^2\right)\Big|_4^6 + \left(\frac{x^3}{2} - 12x^2 + 96x\right)\Big|_6^8\right]$$

$$1 \times \delta = \frac{1}{EI}\left[1.33 + 4.67 + 7.33 + 4\right] = \boxed{\frac{17.33}{EI} = \frac{17.33}{600} = 0.029\,m = 2.9\,cm}$$



$$\frac{1}{2EI}\left(\frac{2}{6}\right)\left(4\times2 + 4\times5\times\frac{3}{2} + 6\times1\right)$$

$$1 \times \delta = \int \frac{mM}{EI}\,dx = \frac{1}{EI}\left(\frac{2}{6}\right)(2)(1) + \frac{1}{2EI}\left(\frac{2}{6}\right)\left[2(2)(1) + 2(4)(2) + (2)(2) + (4)(11)\right]$$

$$+ \frac{1}{2EI}\left(\frac{2}{6}\right)\left[2(4)(2) + 2(6)(1) + (4)(1) + (6)(2)\right] + \frac{1}{EI}\left(\frac{2}{3}\right)(6)(1)$$

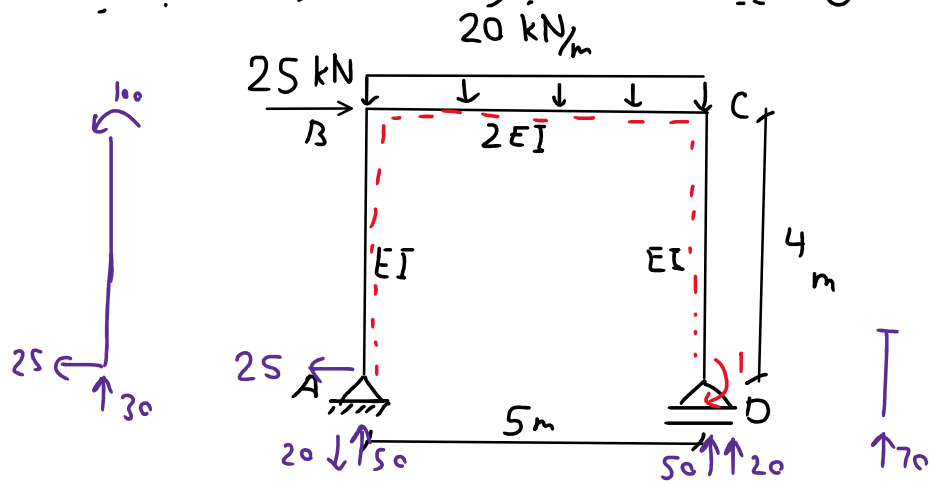
$$1 \times \delta = \frac{1}{EI}\left[1.33 + 4.67 + 7.33 + 4\right] = \boxed{\frac{17.33}{EI} = 2.9\,cm}$$

Deflection Energy4

Wednesday, November 29, 2023 14:04

مثال: تغییرمکان افقی و چرخش نقطه D را محاسبه کنید.

$$1 \times \delta_D = \int \frac{mM}{EI}\,dx$$

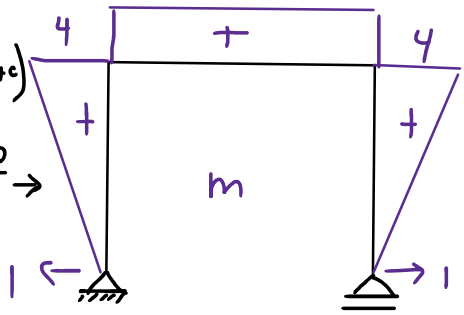


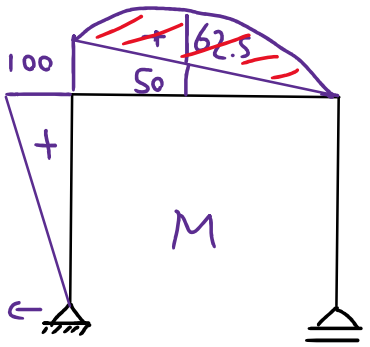
$$1 \times \delta_D = \frac{1}{EI}\left(\frac{4}{3}\right)(4)(100) +$$

$$\frac{1}{2EI}\left(\frac{5}{6}\right)\left((4)(100) + 4(4)(112.5) + 0\right)$$

$$+\ 0 = \frac{1}{EI}(537.3 + 916.7) = \frac{1450}{EI} \rightarrow$$

$$\boxed{\delta_D = \frac{1450}{EI} \rightarrow}$$

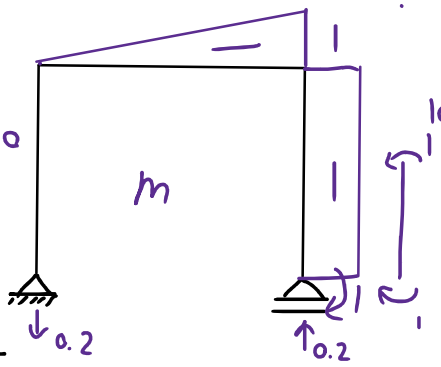


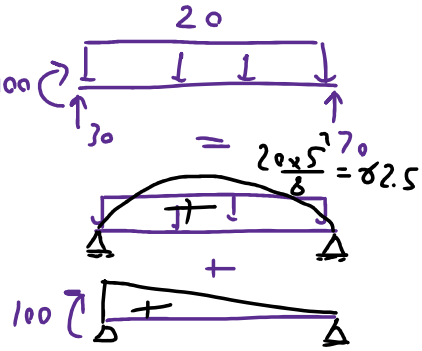


$$1 \times \theta_b = \int \frac{mM}{EI}\,dx$$

$$1 \times \theta_D = 0 + \frac{1}{2EI}\,\frac{5}{6}\left[0 + 4(-0.5)(112.5) + 0\right] + 0$$

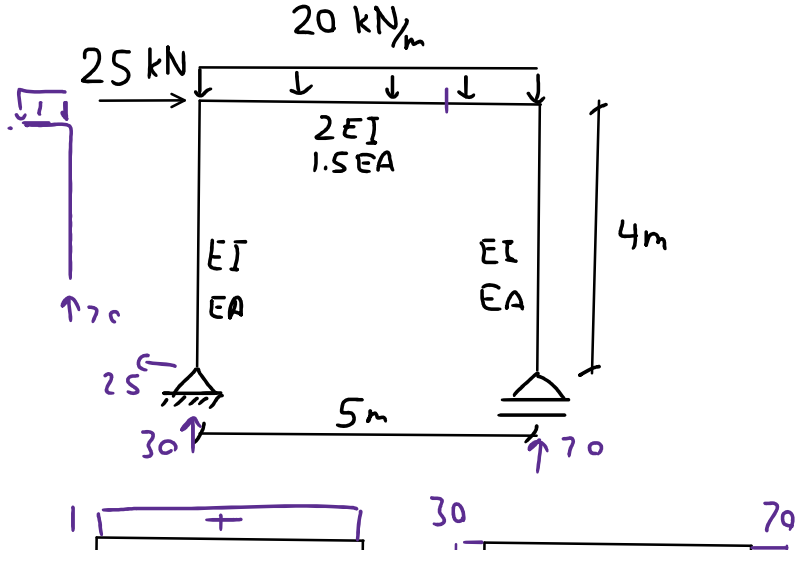
$$\boxed{\theta_D = \frac{-93.75}{EI}\ \curvearrowleft}$$





مثال: مثال فوق را با درنظرگرفتن تغییرشکل ها مجددا حل کنید.

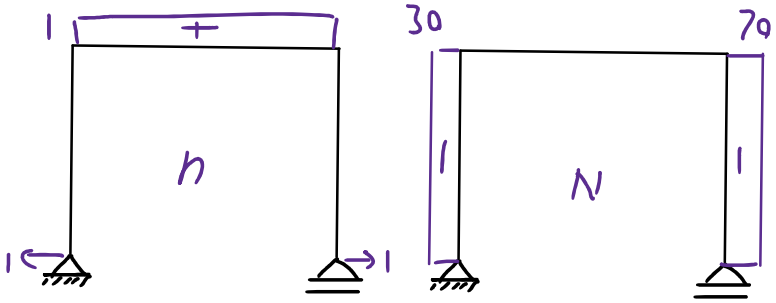
$$1 \times \delta_D = \int \frac{mM}{EI}\,dx + \sum \frac{n_i N_i}{E_i A_i} L_i$$



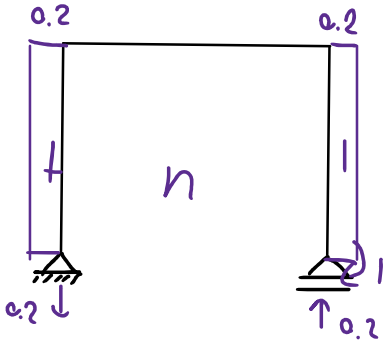
$$* \sum \frac{n_i N_i L_i}{EA} = 0$$

$$\frac{\sum n_i N_i L_i}{EA}$$

$$\Delta_D = \frac{1450}{EI}$$



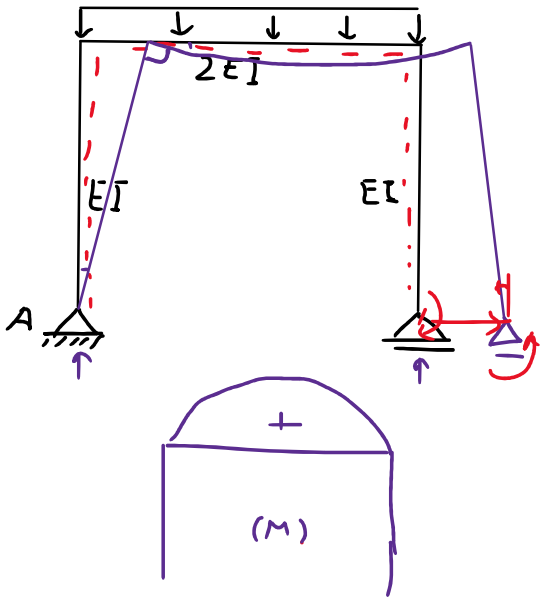
$$1 \times \theta_D = \int \frac{mM}{EI}\,dx + \sum \frac{n_i N_i}{E A_i} L_i$$

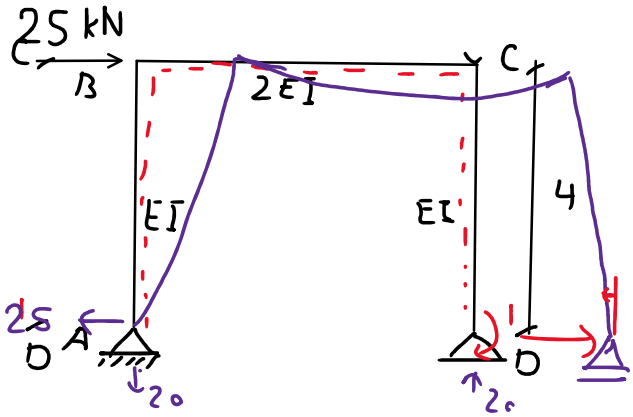


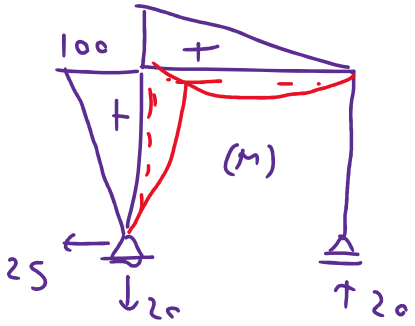
$$\sum \frac{n_i N_i}{E A_i} L_i = \frac{1}{EA}(0.2)(-30)(4) + 0$$

$$+ \frac{1}{EA}(-0.2)(-70)(4) = \frac{32}{EA}$$

$$\boxed{\theta_D = -\frac{93.75}{EI} + \frac{32}{EA}}$$





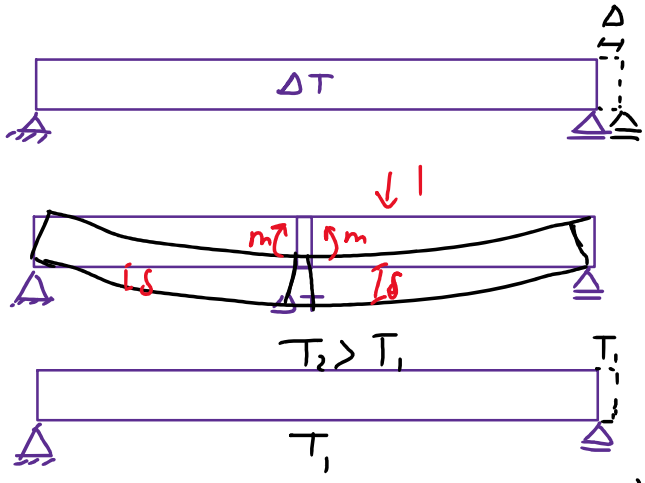


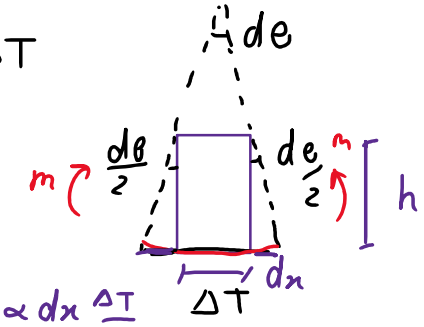
Deflection Energy5

Monday, December 4, 2023 16:47

تغییر شکل خمشی تیر ها ناشی از تغییرات غیر یکنواخت دما

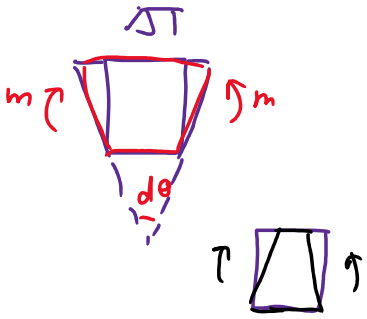
تغییر یکنواخت دما در مقطع تیر شکل زیر، فقط موجب انبساط و انقباض محوری تیر می شود ولی تغییر دما تنها در یک طرف تیر (از بالا یا از پایین)، موجب تغییر شکل خمشی تیر می شود.





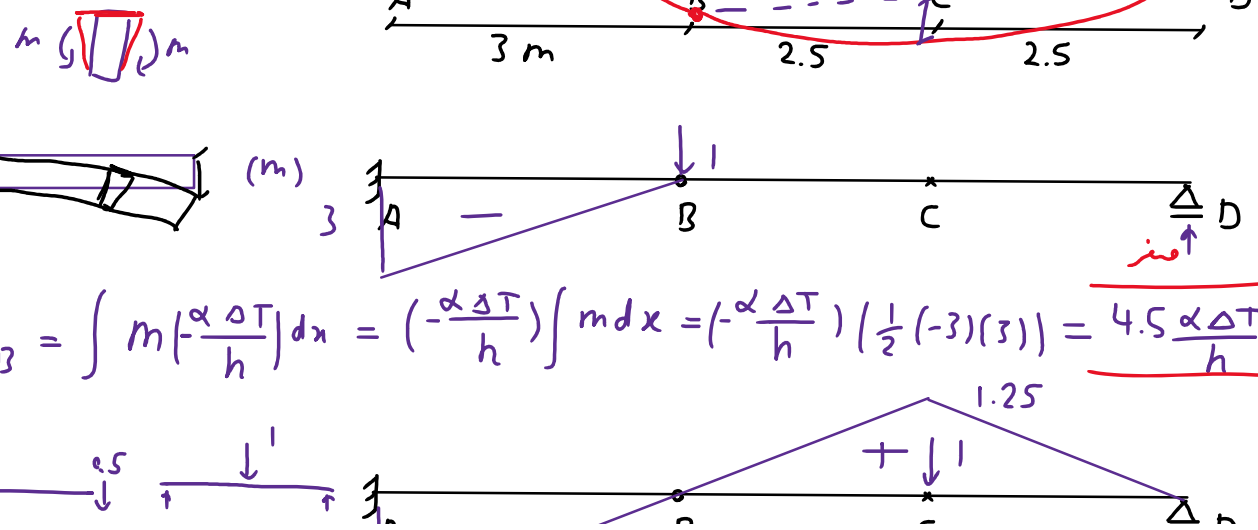
$$\frac{d\theta}{2} = \frac{\alpha\, dx\, \frac{\Delta T}{2}}{h}$$

$$1 \times \delta = \int_0^L m\, d\theta \longrightarrow \boxed{1 \times \delta = \int_0^L m\, \frac{\alpha \Delta T}{h}\, dx}$$

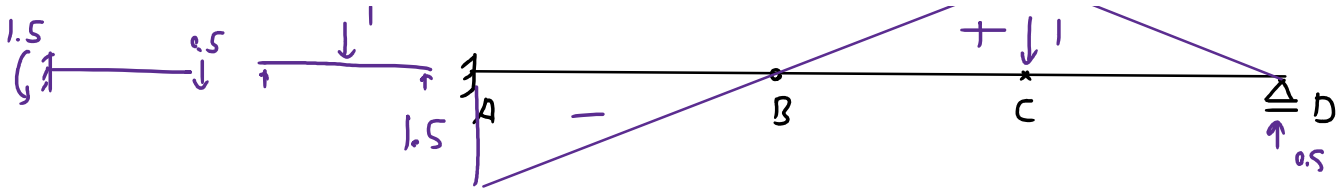


* افزایش دما در تار پایین موجب تغییر شکل خمشی مثبت می شود.

مثال: تغییر مکان قائم نقاط B و C در اثر تغییر دما در تار بالای تیر را به دست آورید. ($\Delta T > 0$)

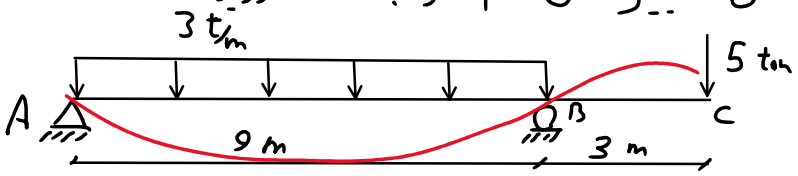


$$1 \times \delta_B = \int m \left(-\frac{\alpha \Delta T}{h}\right) dx = \left(-\frac{\alpha \Delta T}{h}\right) \int m\, dx = \left(-\frac{\alpha \Delta T}{h}\right)\left(\frac{1}{2}(-3)(3)\right) = \boxed{4.5\, \frac{\alpha \Delta T}{h} \downarrow}$$



$$1 \times \delta_c = \int m \frac{\alpha \Delta T}{h} dx = \left(-\frac{\alpha \Delta T}{h}\right)\left(\frac{1}{2}(-1.5)(3)\right) + \left(\frac{\alpha \Delta T}{2h}\right)\left(\frac{1}{2}(1.25)(5)\right) = 3.81 \frac{\alpha \Delta T}{h} \downarrow$$

مثال: تغییر مکان قائم C را به دست آورید. $(EI = 18000\ t.m^2)$

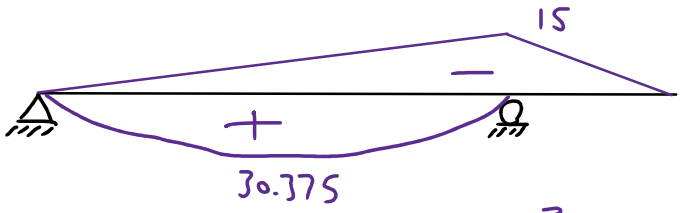


$$1 \times \delta_c = \int \frac{m M}{EI} dx$$

$$1 \times \delta_c = \frac{1}{EI}\left[\left(\frac{9}{3}\right)(-15)(-3) + \left(\frac{9}{3}\right)(-3)(30.375) + \left(\frac{3}{3}\right)(-3)(-15)\right] = \frac{-93.375}{EI}$$

-138.4

(M)

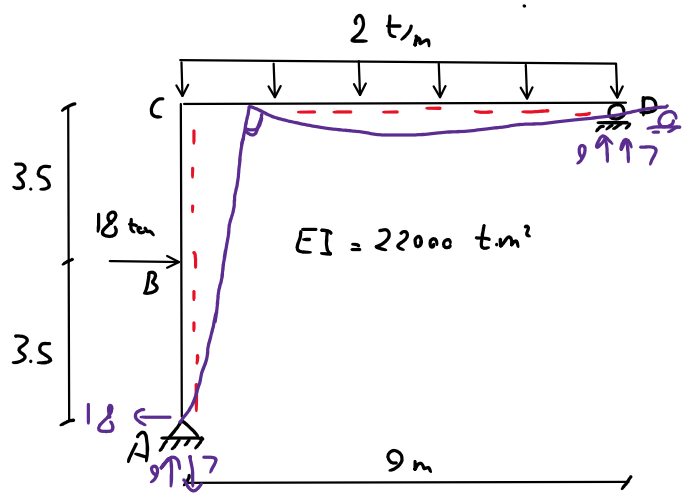


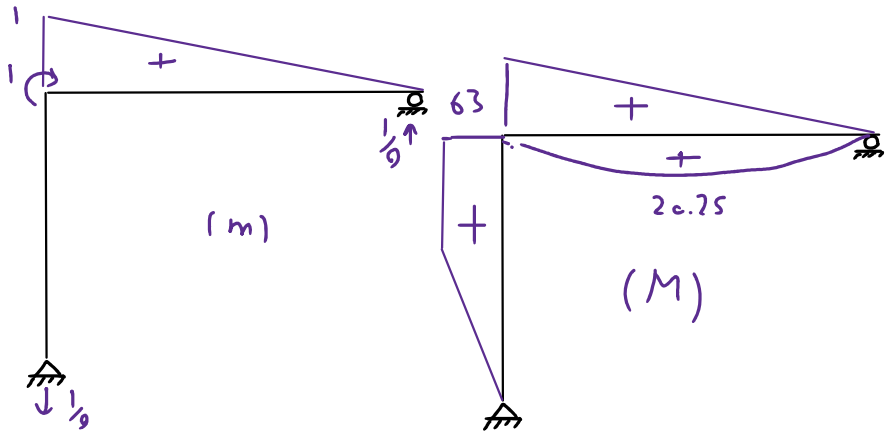
$$\left(\frac{9}{6}\right)\left(0 + 4(-1.5)(22.875) + (-3)(-15)\right) = -138.4$$

$$\delta_c = \frac{-93.375}{18000} = -5.2 \times 10^{-3}\ (m) = 5.2\ mm \uparrow$$

مثال: مطلوب است چرخش گره C.

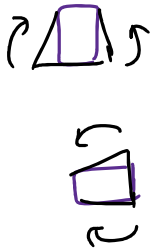
$$1 \times \theta_C = \int \frac{m\,M}{EI}\,dx$$

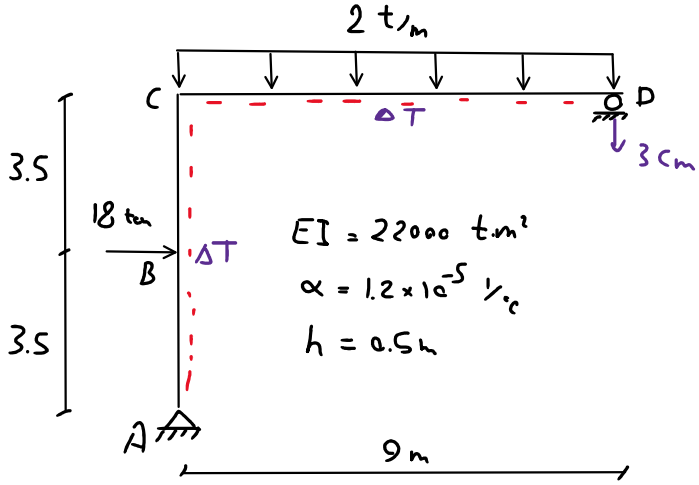


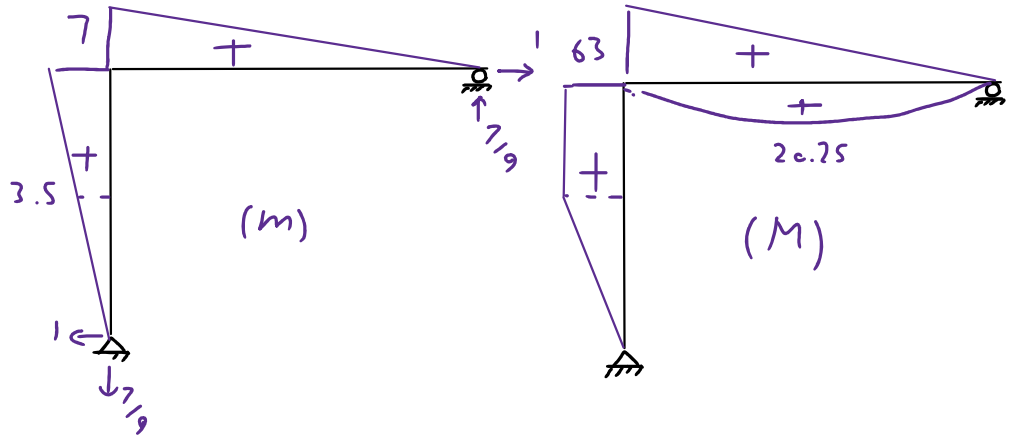


$$1 \times \theta_C = \frac{1}{EI}\left[\left(\frac{9}{3}\right)(1)(63) + \left(\frac{9}{3}\right)(1)(20.25)\right] = \frac{249.75}{EI} = \frac{249.75}{22000} = 0.011\ rad)$$

مثال: مطلوب است تغییر مکان افق D ناشی از بارگذاری خارجی، افزایش دمای داخل تیاب به مقدار 30° و نشست تکیه‌گاه D به مقدار 3 cm.







$$1 \times \delta_D - \left(\frac{7}{9}\right)(0.03) = \int \frac{mM}{EI}dx + \int m\frac{\alpha \Delta T}{h}dx$$

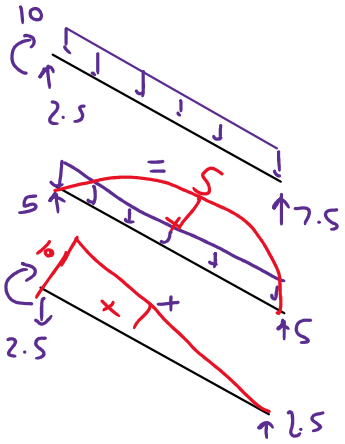
$$\int \frac{mM}{EI}dx = \frac{1}{EI}\left[\left(\frac{9}{3}\right)(7)(63) + \left(\frac{9}{3}\right)(7)(20.25) + \left(\frac{3.5}{3}\right)(3.5)(63) + 63\left(\frac{7+3.5}{2}\right)(3.5)\right]$$

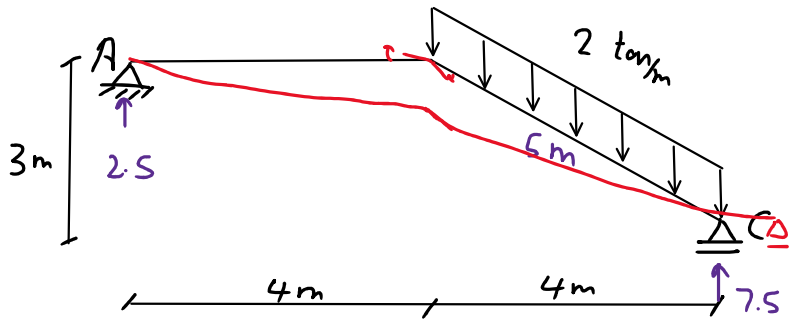
$$\frac{1}{EI}\left(\frac{1748.25}{EI} + \frac{1414.87}{EI}\right) = \frac{3163}{EI} = \frac{3163}{22000} = 0.14\,m = 14\,cm \rightarrow$$

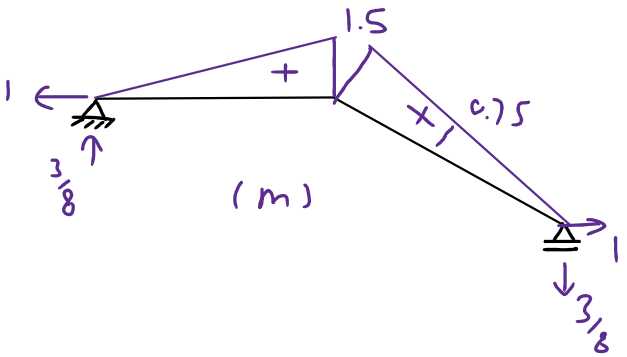
$$\int m\frac{\alpha \Delta T}{h}dx = \frac{\alpha \Delta T}{h}\left[\frac{7\times 9}{2} + \frac{7\times 7}{2}\right] = 58\frac{\alpha \Delta T}{h} = 58 \times \frac{1.2\times 10^{-5} \times 30}{0.5} = 0.04\,m = \overrightarrow{4cm}$$

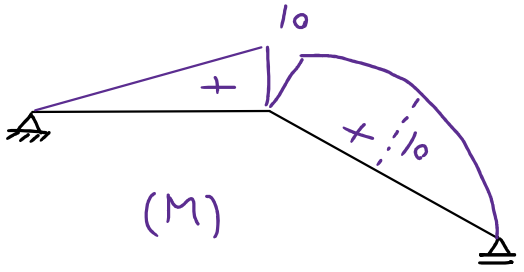
$$1 \times \delta_D - 0.0233 = 0.14 + 0.04 \rightarrow \boxed{\delta_D = 20.33\,\overrightarrow{cm}}$$

مثال: مطلوب است تغییر مکان افق C و تغییر مکان قائم D و چرخش B.



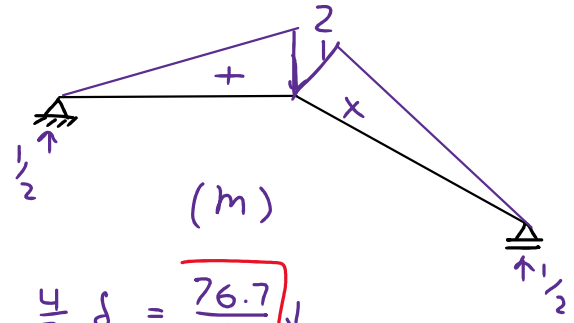




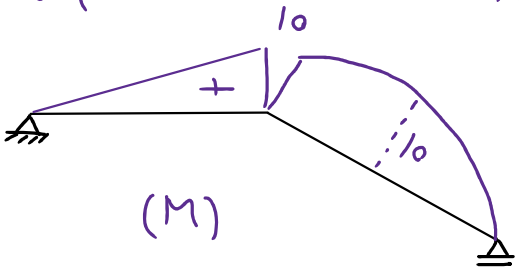


3/8

$$1 \times \delta_C = \int \frac{m M}{EI} dx = \frac{1}{EI}\left[\left(\frac{4}{3}\right)(1.5)(10) + \left(\frac{5}{6}\right)\left((1.5)(10) + 4(0.75)(10) + 0\right)\right] = \frac{57.5}{EI} \rightarrow$$

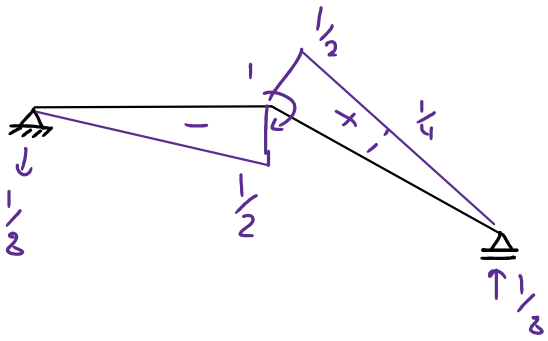


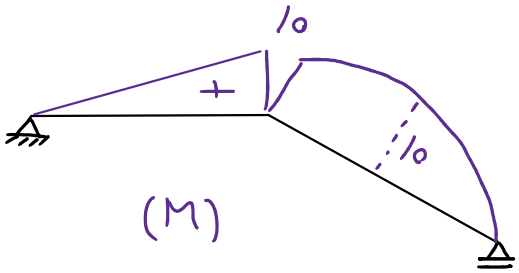
(m)



(M)

$$\delta_B = \frac{4}{3}\delta_C = \frac{76.7}{EI} \downarrow$$





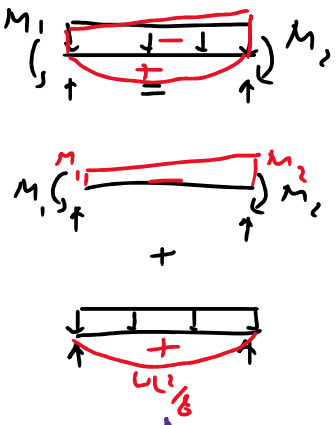
(M)

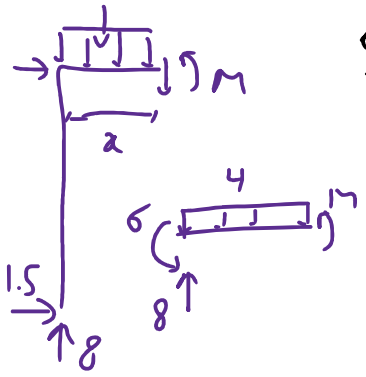
$$1 \times \theta_B = \frac{1}{EI}\left[\left(\frac{4}{3}\right)\left(-\frac{1}{2}\right)(10) + \left(\frac{5}{6}\right)\left(\left(\frac{1}{2}\right)(10) + 4\left(\frac{1}{4}\right)(10) + 0\right)\right] = \frac{5.83}{EI} \circlearrowright$$

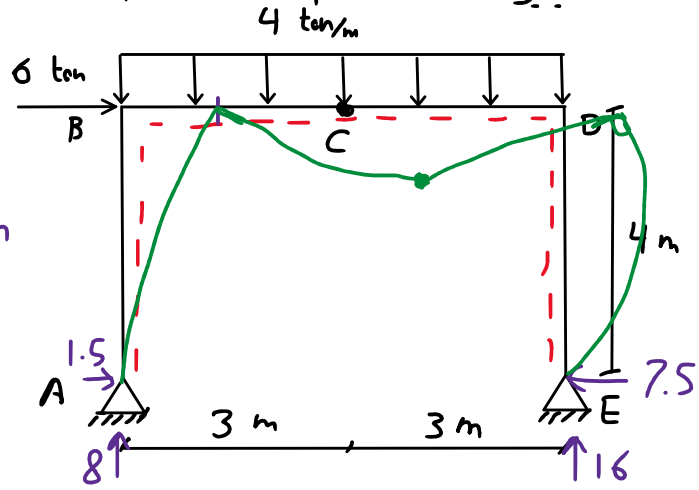
Deflection Energy7

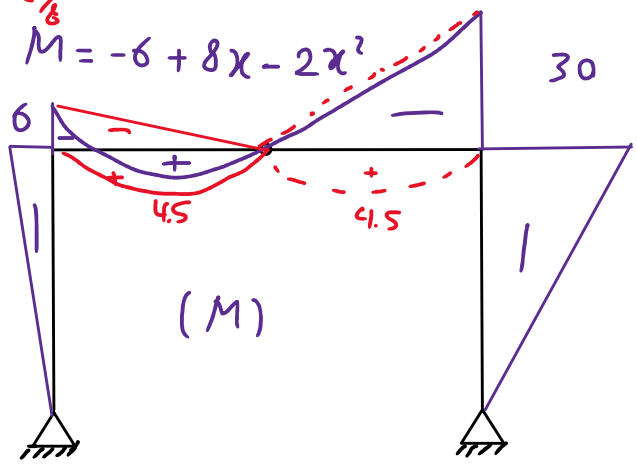
Wednesday, December 6, 2023 9:32

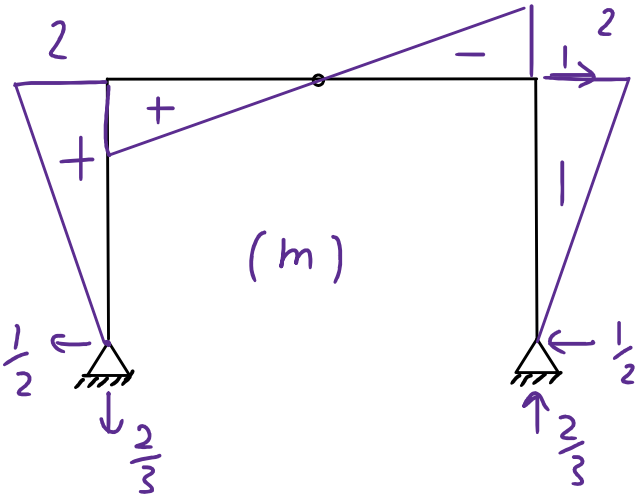
مثال: تغییر مکان افقی و چرخش D ، چرخش سمت چپ مفصل C و Δe_C







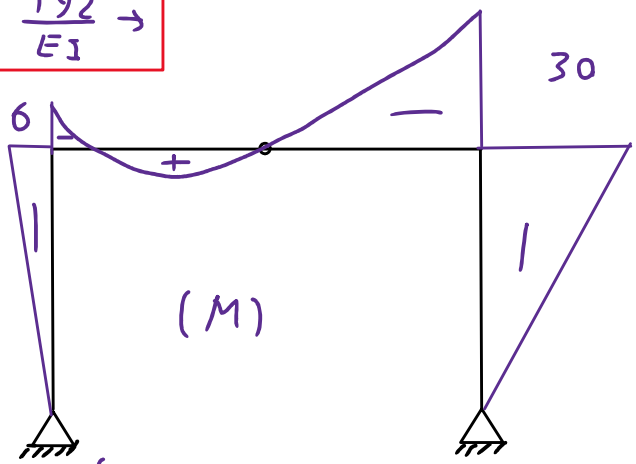


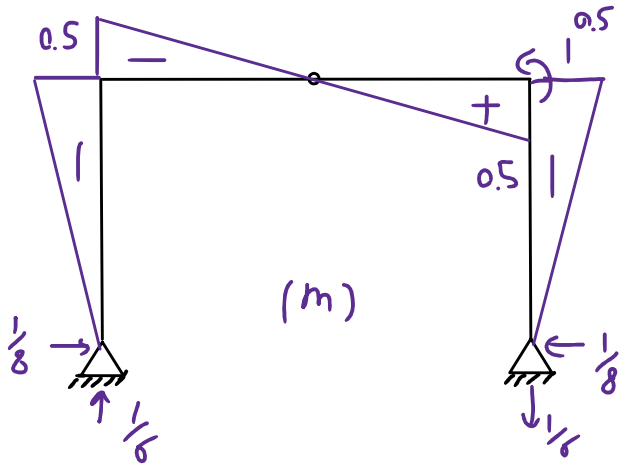


$$1 \times \delta_D = \int \frac{mM}{EI} dx$$

$$EI\delta_D = \left(\frac{4}{3}\right)(2)(-6) + \left(\frac{4}{3}\right)(-2)(-30) + \left(\frac{6}{8}\right)\left[(2)(-6) + 0 + (-2)(-30)\right] = 192$$

$$\delta_D = \frac{192}{EI} \rightarrow$$

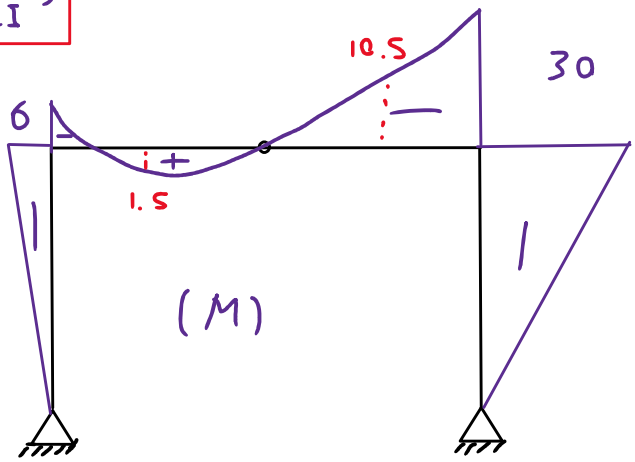


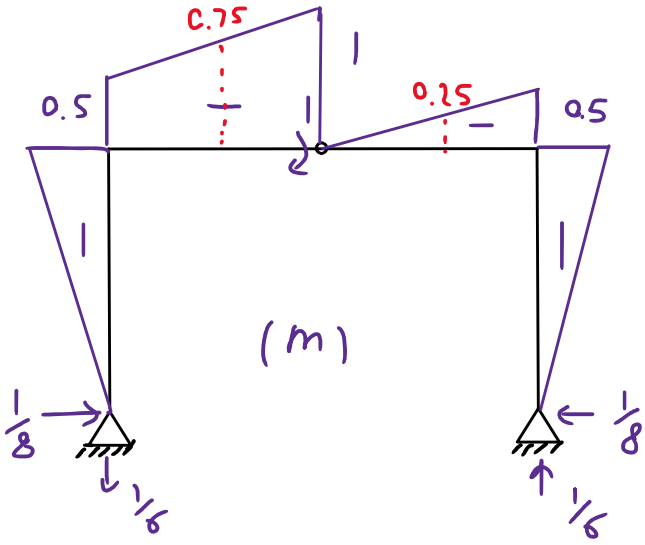


$$1 \times \theta_D = \int \frac{mM}{EI} dx$$

$$EI\theta_D = \left(\frac{4}{3}\right)(-0.5)(-6) + \left(\frac{4}{3}\right)(-0.5)(-30) + \left(\frac{6}{6}\right)\left[(-0.5)(-6) + 0 + (0.5)(-30)\right] = 12$$

$$\theta_D = \frac{12}{EI}\circlearrowleft$$





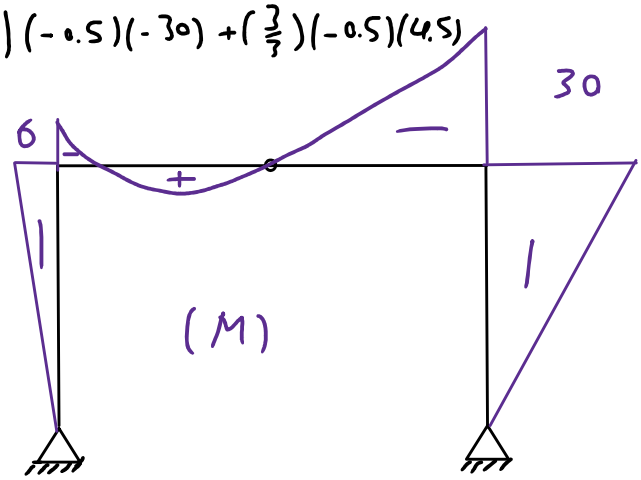
$$1 \times \theta_{CL} = \int \frac{mM}{EI}\,dx$$

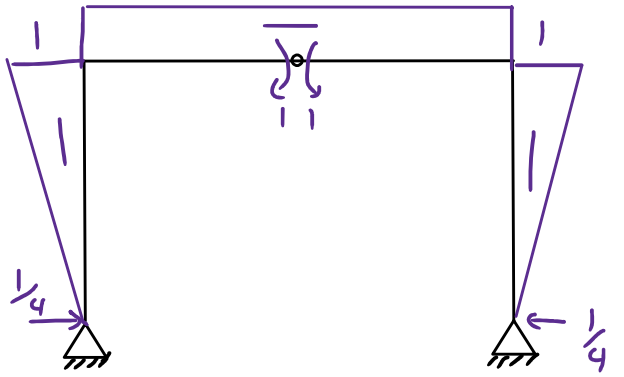
$$EI\theta_{CL} = \underset{4}{\left(\frac{4}{3}\right)}(-0.5)(-6) + \underset{20}{\left(\frac{4}{3}\right)}(-0.5)(-30) + \underset{-0.75}{\left(\frac{3}{6}\right)}\left[(-0.5)(-6) + 4(-0.75)(1.5) + 0\right]$$

$$+\left(\frac{3}{6}\right)\left[0 + 4(-0.25)\underset{12.75}{(-10.5)} + (-0.5)(-30)\right] = 36$$

$$\theta_{CL} = \frac{36}{EI}\circlearrowright$$

$$\left(\frac{3}{5}\right)(-0.5)(-30) + \left(\frac{3}{5}\right)(-0.5)(4.5)$$

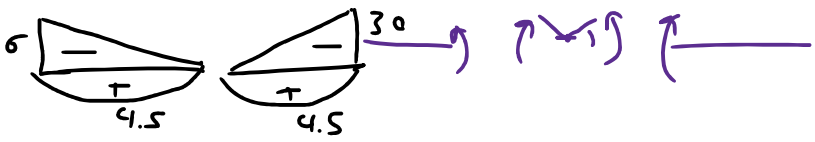




$$1 \times \Delta\theta_C = \int \frac{mM}{EI}\,dx$$

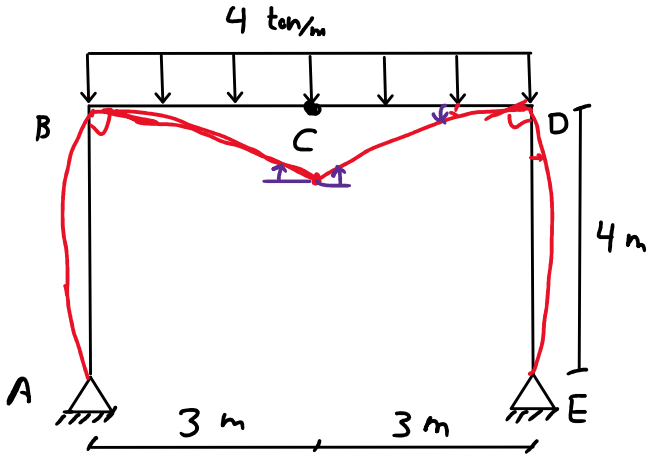
$$EI\,\Delta\theta_C = \underset{8}{\left(\frac{4}{3}\right)}(-1)(-6) + \underset{40}{\left(\frac{4}{3}\right)}(-1)(-30) + \underset{36}{\left(\frac{6}{6}\right)}\left[(-1)(-6) + 0 + (-1)(-30)\right] = 84$$

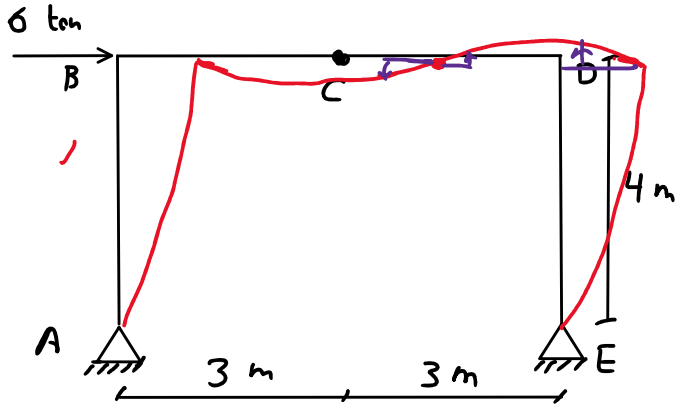
$$\Delta\theta_C = \frac{84}{EI}\uparrow\downarrow\uparrow$$

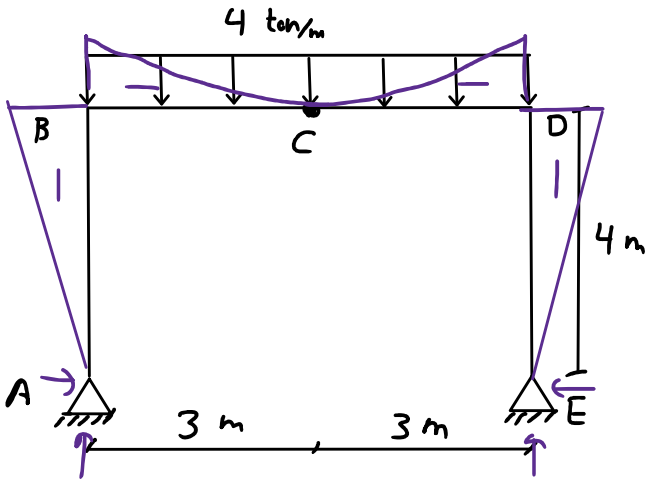


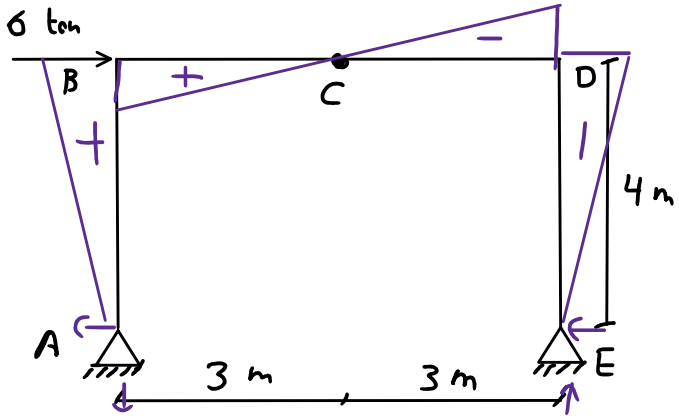
$$\frac{1}{2}(-6)(3) + \frac{1}{2}(-30)(3) + 2 \times \frac{2}{3} \times 4.5 \times 3 = 36$$

$$-9 \qquad -45 \qquad +18$$


4 ton/m
B
C
D
4 m
A
3 m
3 m
E


6 ton
B
C
D
4 m
A
3 m
3 m
E


4 ton/m
B
C
D
4 m
A
3 m
3 m
E


6 ton
B
C
D
4 m
A
3 m
3 m
E

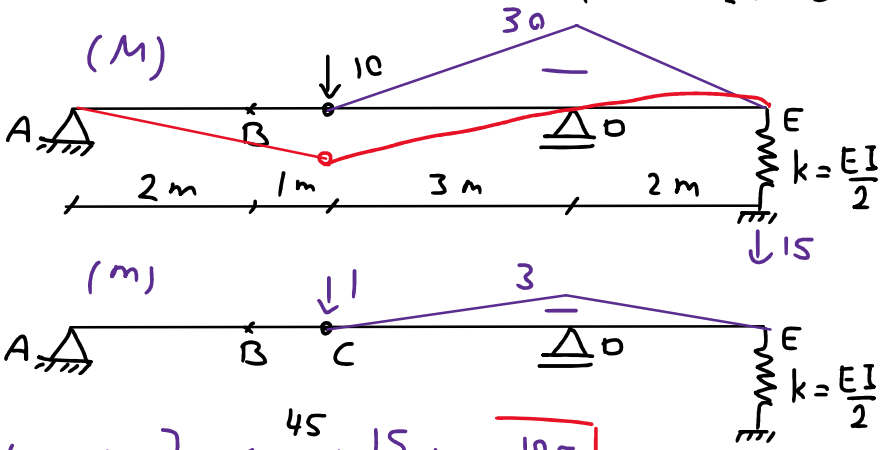
Deflection Energy8

Sunday, December 10, 2023 13:05

$$1 \times \delta = \int \frac{mM}{EI} dx + \sum f \frac{F}{k} + \sum m \frac{M}{k_\theta}$$

مثال: تغییرمکان قائم B را به دست آورید.

$$1 \times \delta_C = \int \frac{mM}{EI} dx + f \frac{F}{k}$$



$$1 \times \delta_C = \frac{1}{EI}\left[\left(\frac{3}{3}\right)(-30)(-3) + \left(\frac{2}{3}\right)(-30)(-3)\right] + (1.5)\left(\frac{15}{EI/2}\right) = \frac{195}{EI}$$

(150, 45)

مثال: تغییرمکان زیربار را به دست آورید.

بار واحد

$$1 \times \delta_B = \int \frac{mM}{EI} dx = 0 + \left(\frac{L}{6}\right)\left(\frac{L}{4}\right)\left(\frac{PL}{4EI}\right) = \frac{PL^3}{96EI}$$

سطح لنگ

$$\delta_{C/A} = \frac{1}{2}\left(\frac{PL}{4EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right) = \frac{PL^3}{48EI}$$

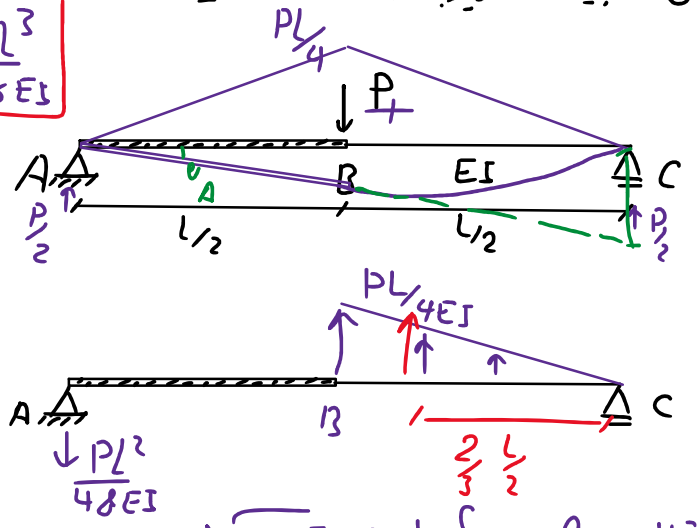
$$\theta_A = \frac{\delta_{C/A}}{L} = \frac{PL^2}{48EI}$$

$$\delta_B = \theta_A\left(\frac{L}{2}\right) = \frac{PL^3}{96EI}$$

تیرمزدوج

$$\sum M_C = 0 \rightarrow R_A L - \frac{1}{2}\left(\frac{PL}{4EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right) = 0$$

$$M_B = \left(\frac{PL^2}{48EI}\right)\left(\frac{L}{2}\right) = \frac{PL^3}{96EI}$$

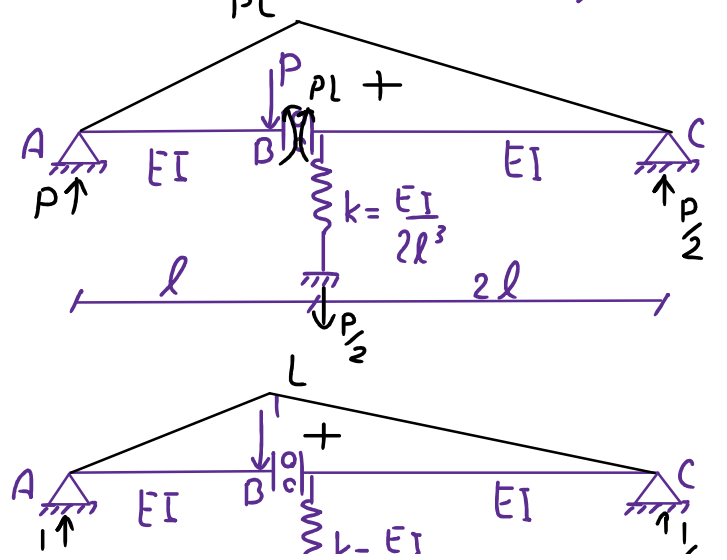


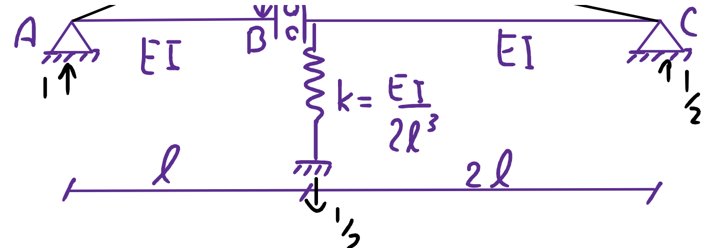
مثال: θ_A ، δ_{BL} را به دست آورید.

$$1 \times \delta_{BL} = \int \frac{mM}{EI} dx + f\left(\frac{F}{k}\right)$$

$$1 \times \delta_{BL} = \frac{1}{EI}\left[\left(\frac{3L}{3}\right)(L)(PL)\right] + \left(\frac{1}{2}\right)\left(\frac{P/2}{\frac{EI}{2L^3}}\right) =$$

$$\delta_{BL} = \frac{3}{2}\frac{PL^3}{EI}$$

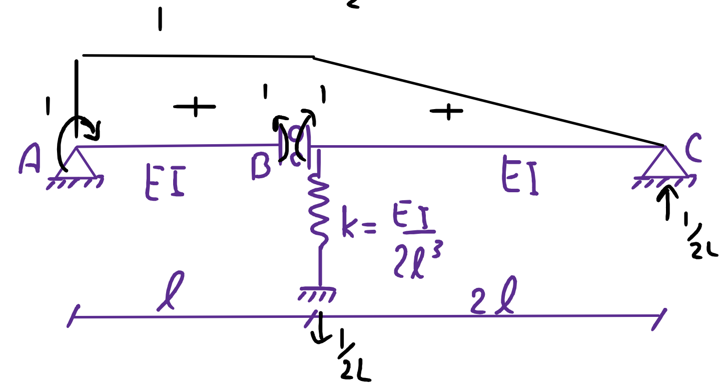




$1\times\theta_A = \int \frac{mM}{EI}dx + f\left(\frac{F}{k}\right)$

$1\times\theta_A = \frac{1}{EI}\left[(1)\frac{1}{2}(PL)(L) + \left(\frac{2L}{3}\right)(1)(PL)\right]$

$+\left(\frac{1}{2L}\right)\left(\frac{P/2}{\frac{EI}{2L^3}}\right) = \frac{5}{3}\frac{PL^2}{EI}$



مثال: تغییر مکان قائم B را بدست آورید.

بار واحد

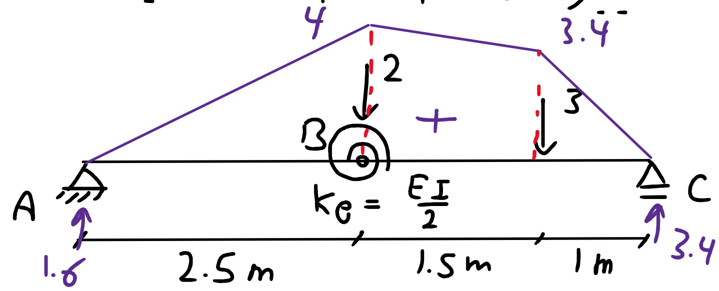
$1\times\delta_B = \int\frac{mM}{EI}dx + m\left(\frac{M}{k_\theta}\right)$

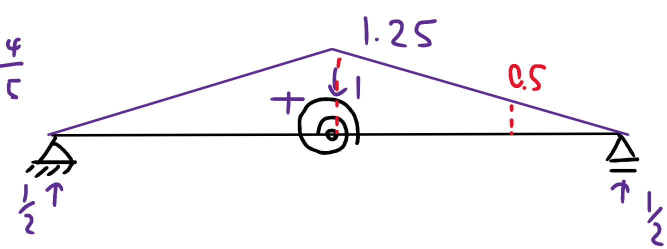
$\delta_B = \frac{1}{EI}\left[\left(\frac{2.5}{3}\right)(1.25)(4) + \left(\frac{1.5}{6}\right)\left((1.25)(4)+4\left(\frac{1.75}{2}\right)(3.7)+(0.5)(3.4)\right) + \left(\frac{1}{3}\right)(0.5)(3.4)\right] + (1.25)\left(\frac{4}{\frac{EI}{2}}\right) = \frac{19.64}{EI}$

9.64

10

$\delta_B = \frac{19.64}{EI}$





لنگر سطح

اصلی δ Δθ

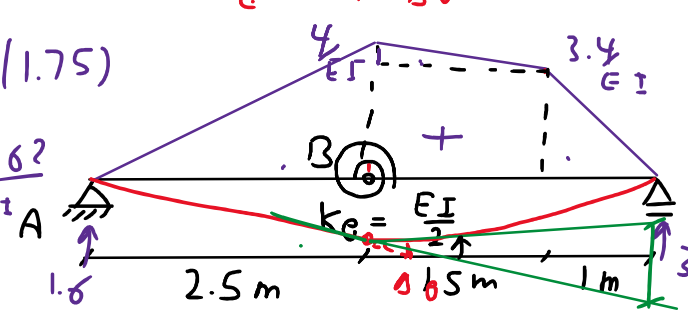
مزدوج M ΔV

$\delta_{C/A} = \left(\frac12\right)\left(\frac{4}{EI}\right)(2.5)(3.333) + \left(\frac{3.4}{EI}\right)(1.5)(1.75)$

$+ \frac12\left(\frac{0.6}{EI}\right)(1.5)(2) + \frac12\left(\frac{3.4}{EI}\right)(1)\left(\frac23\right) = \frac{27.62}{EI}$

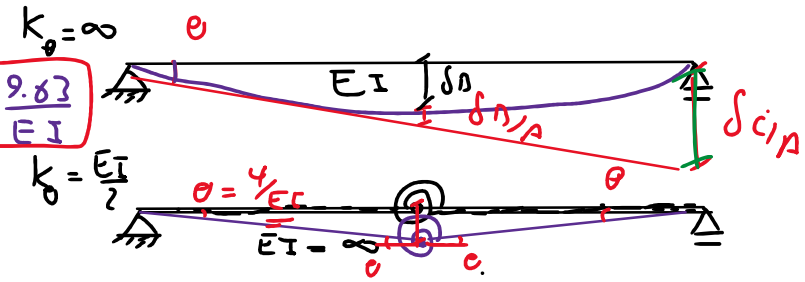
$\theta_A = \frac{\delta_{C/A}}{5} = \frac{5.52}{EI}$

$\delta_B = -2.5\,\theta_A + \delta_{B/A} =$



$$\delta_B = -2.5\,\theta_A + \delta_{B/A} =$$

$$-2.5\left(\frac{5.52}{EI}\right) + \frac{1}{2}\left(\frac{4}{EI}\right)(2.5)\left(\frac{2.5}{3}\right) = \frac{9.63}{EI}$$

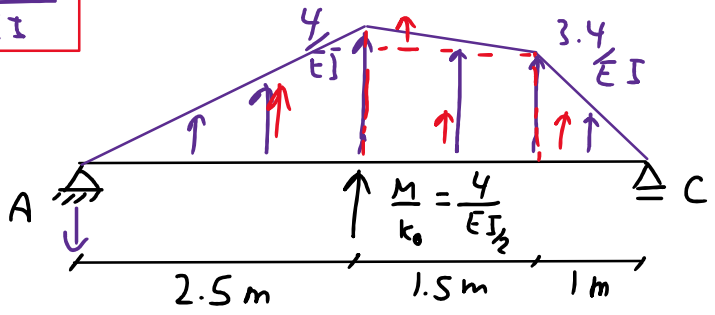


$$\Delta\theta = \frac{M}{k_\theta} = \frac{4}{\frac{EI}{2}} = \frac{8}{EI}$$

$$\delta_B = 2.5\,\theta = 2.5\left(\frac{4}{EI}\right) = \frac{10}{EI}$$

تیر سردوج

$$\delta_D = \frac{19.63}{EI}$$



$$\sum M_C = 0$$

$$\left(\frac{1}{2}\right)\left(\frac{4}{EI}\right)(2.5)(3.333) + \left(\frac{3.4}{EI}\right)(1.5)(1.75)$$

$$+ \frac{1}{2}\left(\frac{0.1}{EI}\right)(1.5)(2) + \frac{1}{2}\left(\frac{3.4}{EI}\right)(1)\left(\frac{2}{3}\right) + \left(\frac{8}{EI}\right)(2.5) = R_A(5) \rightarrow R_A = \frac{9.52}{EI}$$

$$\delta_B = M_B = \frac{-9.52}{EI}(2.5) + \frac{1}{2}\left(\frac{4}{EI}\right)(2.5)\left(\frac{2.5}{3}\right) = \frac{18.64}{EI}$$

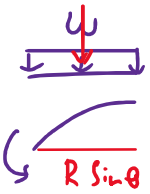
اصل — مزدوج

Deflection Energy9

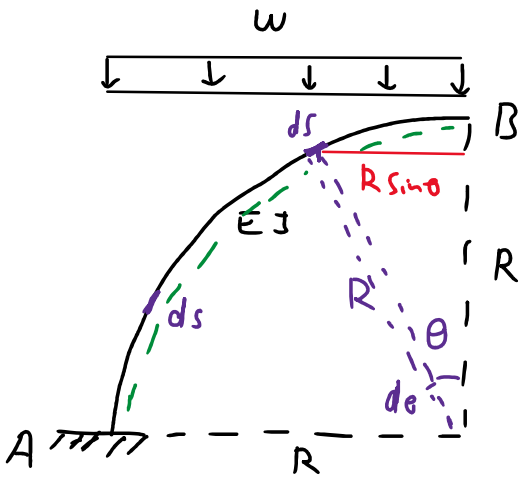
Wednesday, November 29, 2023 16:54

مثال: مطلوب است تغییر مکان قائم و افق نقطه B .

$$1 \times \delta_{By} = \int \frac{mM}{EI} ds$$



$$M = -w\frac{(R\sin\theta)^2}{2}$$

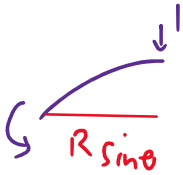


$$1 \times \delta_{By} = \frac{1}{EI}\int_0^{\frac{\pi}{2}} (-R\sin\theta)\left(-\frac{w}{2}R^2\sin^2\theta\right) R\,d\theta$$

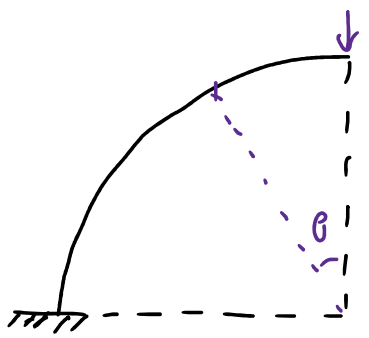
$$\delta_{By} = \frac{wR^4}{2EI}\int_0^{\frac{\pi}{2}} \sin\theta\,(1-\cos^2\theta)\,d\theta$$

$$= \frac{wR^4}{2EI}\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{wR^4}{2EI}\left[0 - \left(-1+\frac{1}{3}\right)\right]$$



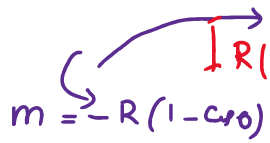
$$m = -R\sin\theta$$



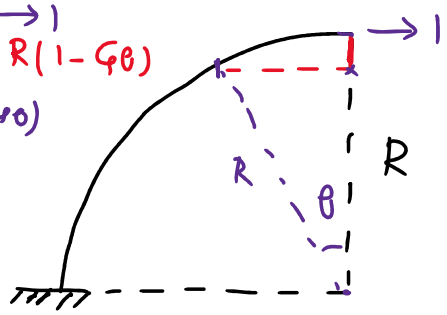
$$\boxed{\delta_{By} = \frac{wR^4}{3EI}}$$

$$1 \times \delta_{Bx} = \int \frac{mM}{EI} ds = \frac{1}{EI}\int_0^{\frac{\pi}{2}} \left(-R(1-\cos\theta)\right)\left(-\frac{w}{2}R^2\sin^2\theta\right) R\,d\theta$$

$$\delta_{Bx} = \frac{wR^4}{2EI}\int_0^{\frac{\pi}{2}} (\sin^2\theta - \cos\theta\sin^2\theta)\,d\theta$$



$$m = -R(1-\cos\theta)$$



$$\delta_{Bx} = \frac{wR^4}{2EI}\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2\theta}{2} - \cos\theta\sin^2\theta\right) d\theta$$

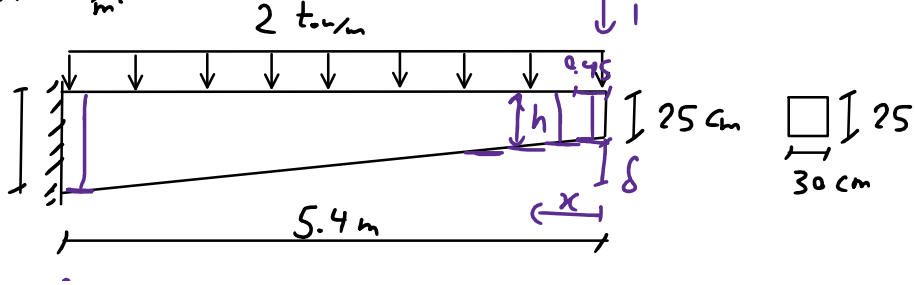
$$\frac{wR^4}{2EI}\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta - \frac{\sin^3\theta}{3}\right]_0^{\frac{\pi}{2}} = \frac{wR^4}{2EI}\left[\left(\frac{\pi}{4} - 0 - \frac{1}{3}\right) - 0\right]$$

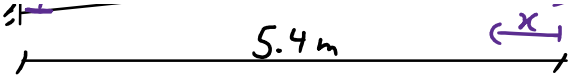
$$\boxed{\delta_{Bx} = 0.226\frac{wR^4}{EI}}$$

مثال: تغییر مکان سر آزاد تیر را به دست آورید.

$$E = 2\times10^6 \ \frac{ton}{m^2}$$

$$h = 0.25 + \frac{0.45}{5.4}x = 0.25 + \frac{1}{12}x$$





$$1 \times \delta = \int \frac{m\,M}{EI}\,dx = \int_{0}^{5.4} \frac{(-x)(-x^2)}{E \times \frac{1}{12} \times 0.3 \times \underbrace{\left(\frac{1}{4} + \frac{1}{12}x\right)^3}_{u}}\,dx$$

$$u = \frac{1}{4} + \frac{1}{12}x$$

$$x = 12\left(u - \tfrac{1}{4}\right)$$

$$dx = 12\,du$$

$$1 \times \delta = \int_{0.25}^{0.7} \frac{12^3\,(u - 1/4)^3}{2 \times 10^6 \times \frac{0.3}{12}\,u^3}\,12\,du =$$

$$\delta = \frac{12^5}{2 \times 10^6 \times 0.3} \int_{0.25}^{0.7} \frac{(u - 1/4)^3}{u^3}\,du = 0.41472 \int_{0.25}^{0.7} \frac{u^3 - \frac{3}{4}u^2 + \frac{3}{16}u - \frac{1}{64}}{u^3}\,du$$

$$\delta = 0.41472 \int_{0.25}^{0.7} \left(1 - \frac{3}{4}\frac{1}{u} + \frac{3}{16}\frac{1}{u^2} - \frac{1}{64}\frac{1}{u^3}\right) du$$

$u^n \qquad u^{-2}$

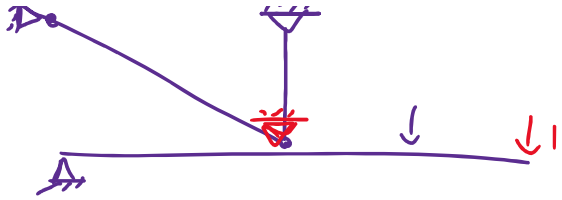
$\frac{1}{n+1}\,u^{n+1}$

$$\delta = 0.41472 \left(u - \frac{3}{4}\ln u + \frac{3}{16}\frac{-1}{u} - \frac{1}{64}\frac{-\frac{1}{2}}{u^2}\right)\Bigg|_{0.25}^{0.7}$$

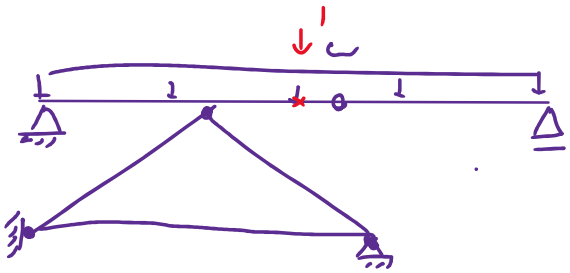
$$\delta = 0.41472 \left[(0.7 - 0.25) - \frac{3}{4}(\ln 0.7 - \ln 0.25) - \frac{3}{16}\left(\frac{1}{0.7} - \frac{1}{0.25}\right) + \frac{1}{128}\left(\frac{1}{0.7^2} - \frac{1}{0.25^2}\right)\right]$$

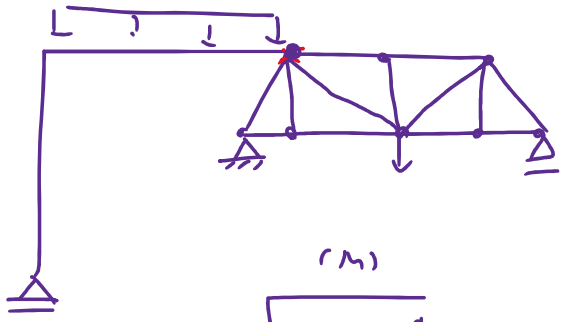
$$\delta = 0.021\ m = 2.1\ Cm$$

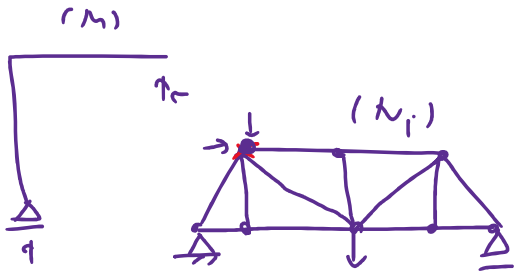
x	h		
0.225	0.26875	0.58682	0.26407
0.675	0.30625	10.7074	4.81832
1.125	0.34375	35.0533	15.774
1.575	0.38125	70.5037	31.7267
2.025	0.41875	113.086	50.8889
2.475	0.45625	159.631	71.8339
2.925	0.49375	207.901	93.5554
3.375	0.53125	256.404	115.382
3.825	0.56875	304.18	136.881
4.275	0.60625	350.633	157.785
4.725	0.64375	395.416	177.937
5.175	0.68125	438.34	197.253
			1054.1
			0.02108



$$1 \times \delta = \int \frac{mM}{EI} dx + \sum \frac{n_i N_i l_i}{EA_i}$$







$$1 \times \delta = \int \frac{mM}{EI} dx + \sum \frac{n_i N_i l_i}{EA_i}$$

Deflection Energy10

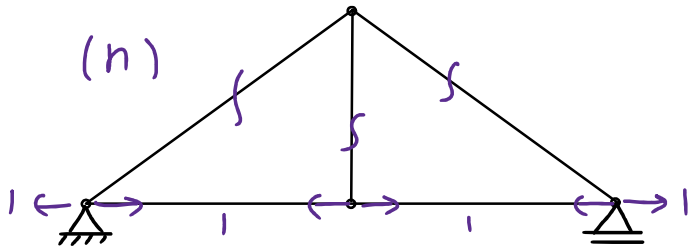
Thursday, December 14, 2023 10:22

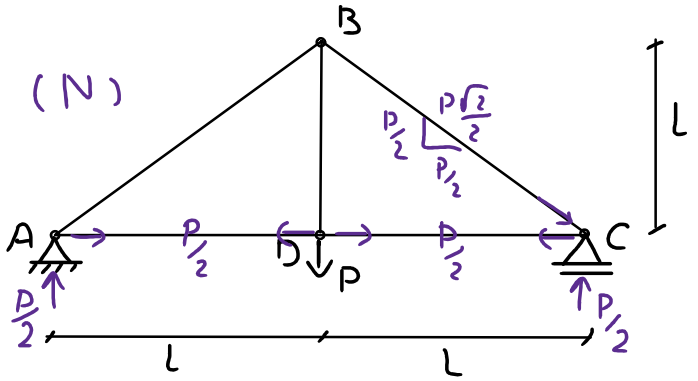
محاسبه تغییر شکل در خرپاها

در اکثر سازه های صنعتی از جمله خرپاها، نیروی محوری و EA در طول عضو ثابت هستند. بنابراین کار داخلی ناشی از بار واحد را می توان به صورت زیر نوشت:

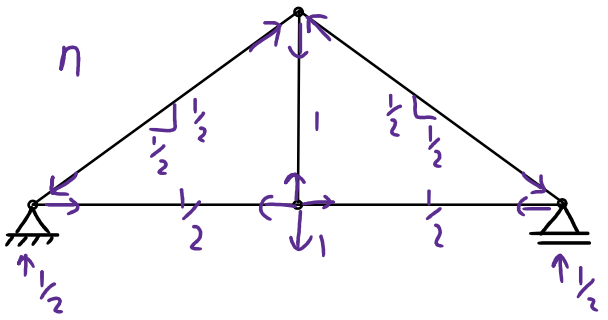
$$1 \times \delta = \sum \frac{n_i N_i l_i}{E_i A_i} \qquad \int \frac{n N}{EA} dx$$

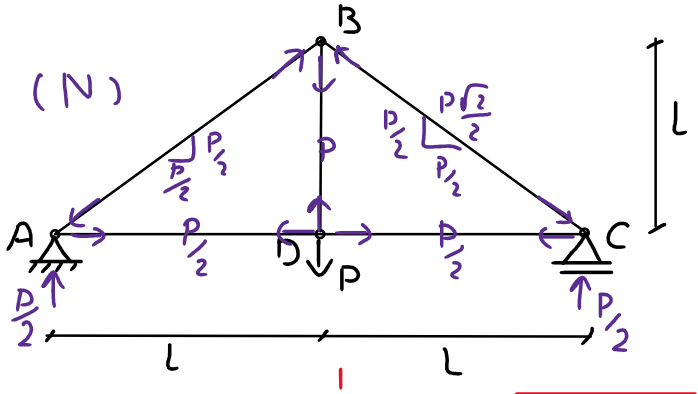
مثال: تغییر مکان افقی گره C و تغییر مکان قائم B و D را به دست آورید.





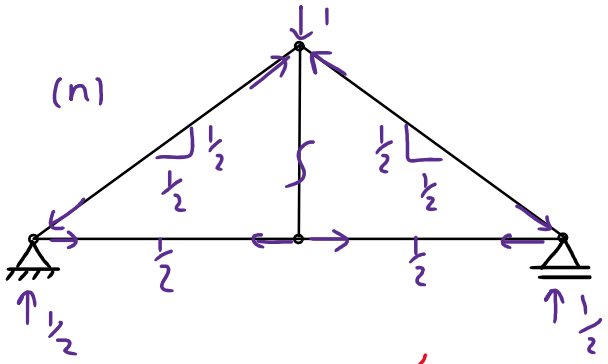
$$1 \times \delta_{Cx} = \frac{1}{EA}(1)\left(\frac{P}{2}\right)(L) \times 2 = \frac{PL}{EA}$$

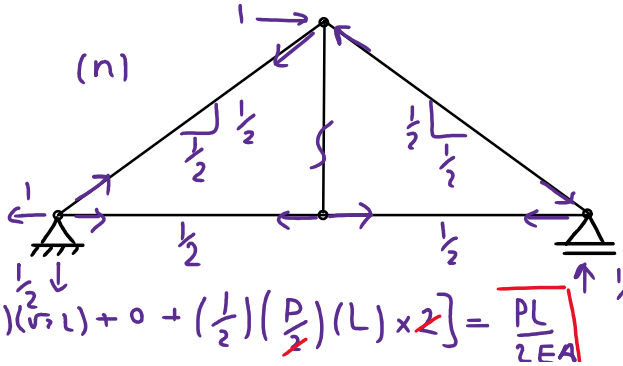




$$1 \times \delta_{Dy} = \frac{1}{EA}\left[\left(\frac{1}{2}\right)\left(\frac{P}{2}\right)(L) \times 2 + \left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}P\right)(\sqrt{2}L) \times 2 + (1)(P)(L)\right] = \left(\frac{3}{2}+\sqrt{2}\right)\frac{PL}{EA}$$

$$\delta_{By} = \left(\frac{1}{2}+\sqrt{2}\right)\frac{PL}{EA}$$



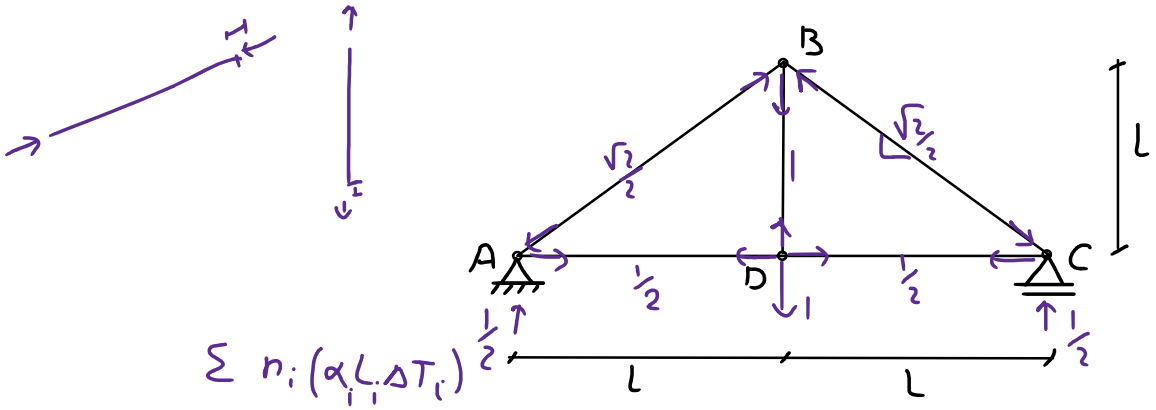


$$1 \times \delta_{Bx} = \frac{1}{EA}\left[\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{P\sqrt{2}}{2}\right)(\sqrt{2}L) + \left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{P\sqrt{2}}{2}\right)(\sqrt{2}L) + 0 + \left(\frac{1}{2}\right)\left(\frac{P}{2}\right)(L) \times 2\right] = \frac{PL}{2EA}$$

$$1\times\delta_{Dx}=\frac{1}{EA}\left[\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{P\sqrt{2}}{2}\right)(\sqrt{2}L)+\left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{P\sqrt{2}}{2}\right)(\sqrt{2}L)+0+\left(\frac{1}{2}\right)\left(\frac{P}{2}\right)(L)\times 2\right]=\frac{PL}{2EA}$$

اثر نشست تکیه‌گاهی، تغییر دما و نقص عضو در خرپاها

مثال: مطلوب است تعیین مکان قائم گره D در اثر نشست تکیه‌گاه C به اندازه 4 cm، افزایش دما در اعضای مایل به اندازه ΔT، اضافه طول عضو BD به مقدار 3 و کوتاه بودن طول عضو CD به اندازه 2 cm.

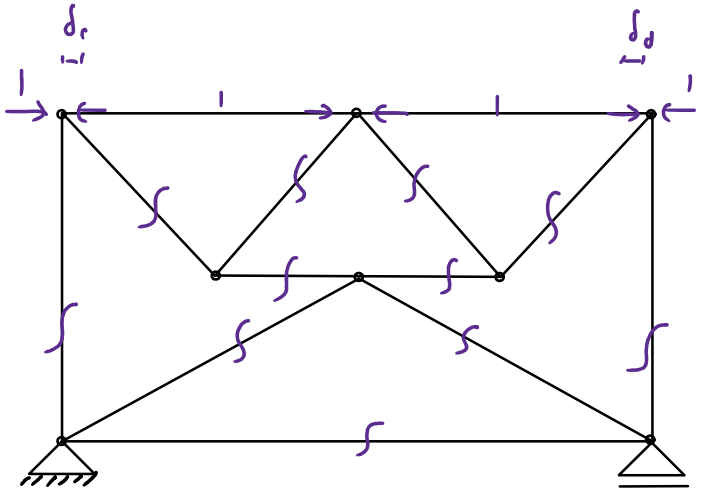


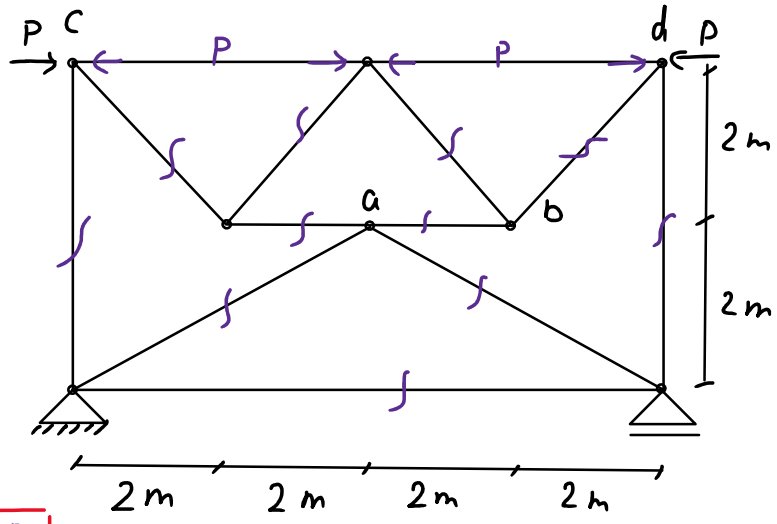
$$\sum n_i(\alpha L_i\Delta T_i)$$

$$1\times\delta-\frac{1}{2}(0.04)=\left(-\frac{\sqrt{2}}{2}\right)(\alpha\sqrt{2}L\Delta T)\times 2+(1)(0.03)+\left(\frac{1}{2}\right)(-0.02)$$

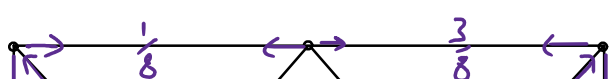
$$\boxed{\delta_{Dy}=0.04+2\alpha L\Delta T}$$

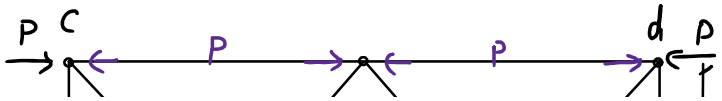
مثال: مطلوب است میزان نزدیک شدن یا دور شدن دو گره C و d و دوران عضو ab.

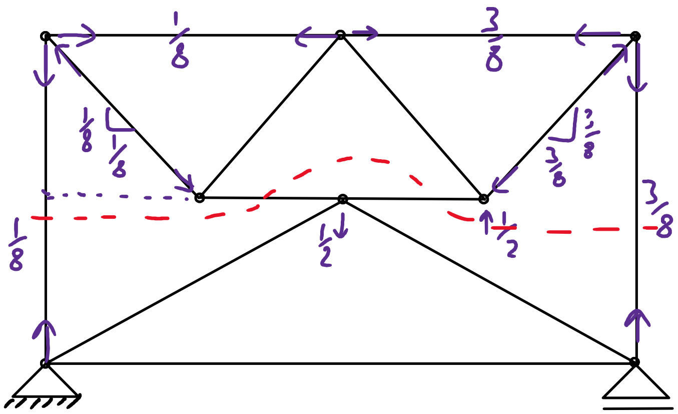


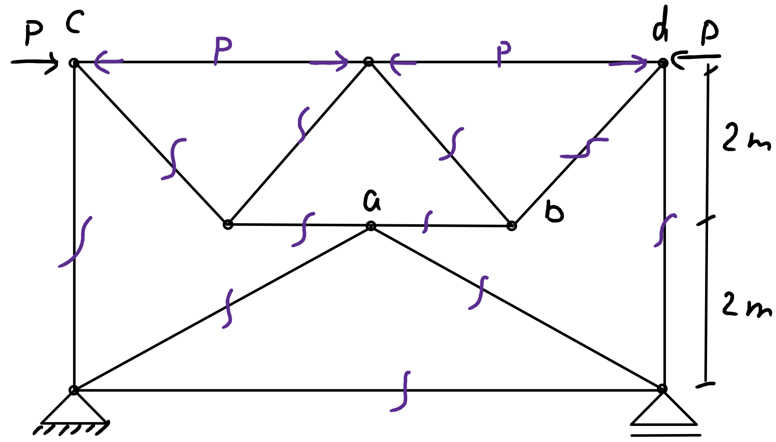


$$\delta_{C/d}=1\times\delta_c+1\times\delta_d=\frac{1}{EA}(-1)(-P)(4)\times 2=\frac{8P}{EA}$$



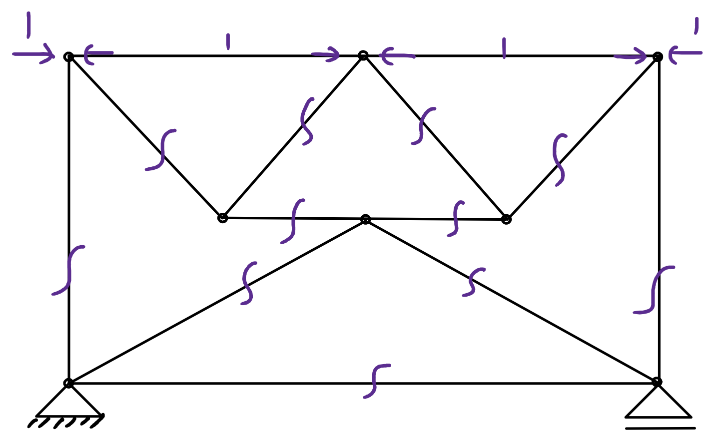


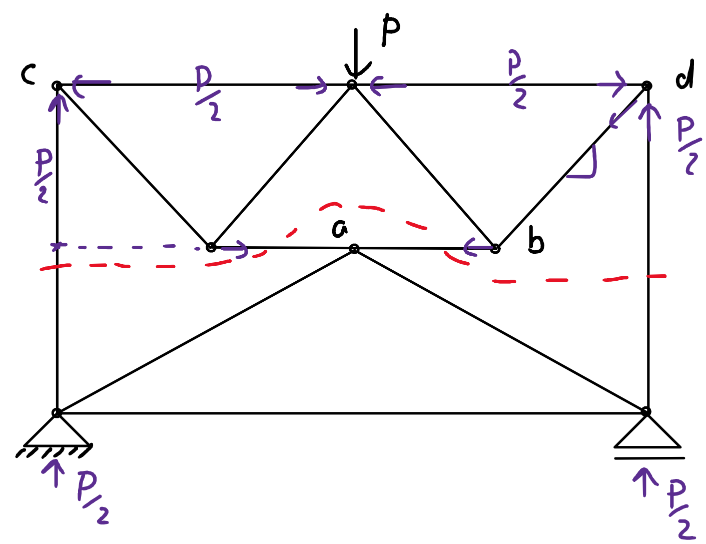




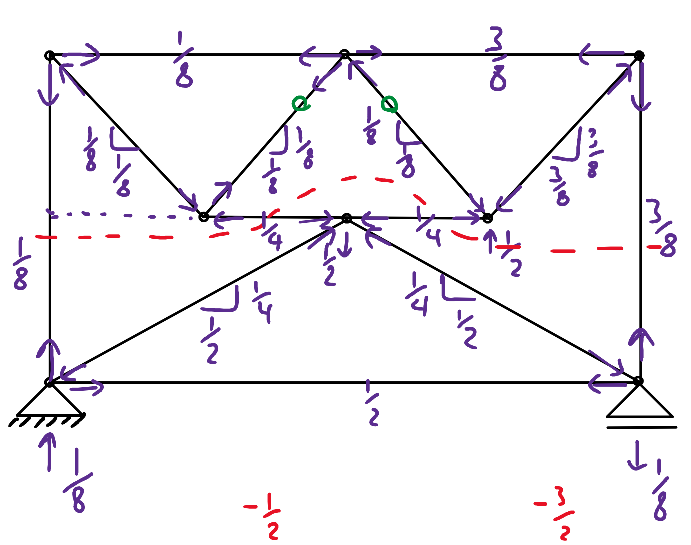
$$\theta_{ab} = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b = \frac{1}{EA}\left[\left(\frac{1}{8}\right)(-P)(4) + \left(\frac{3}{8}\right)(-P)(4)\right] = \boxed{\frac{-2P}{EA}}$$

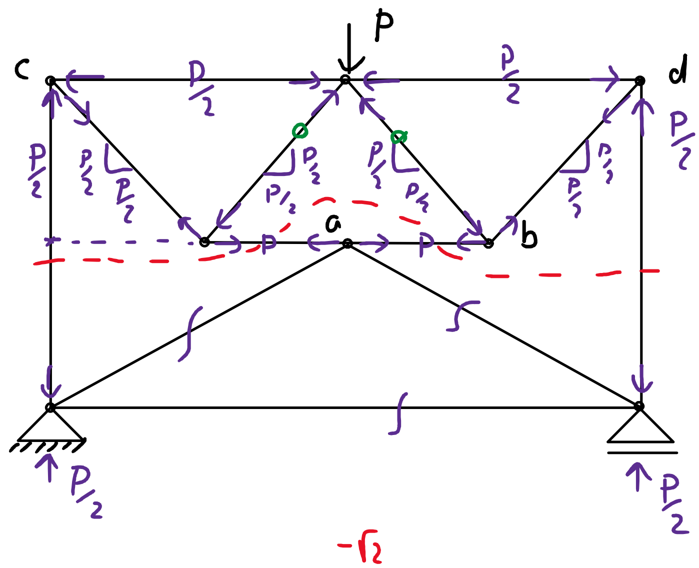
مثال: مطلوب است میزان نزدیک شدن یا دور شدن دو گره c و d و دوران عضو ab.





$$\delta_{c/d} = \frac{1}{EA}(-1)\left(-\frac{P}{2}\right)(4)\times 2 = \boxed{\frac{4P}{EA}}$$



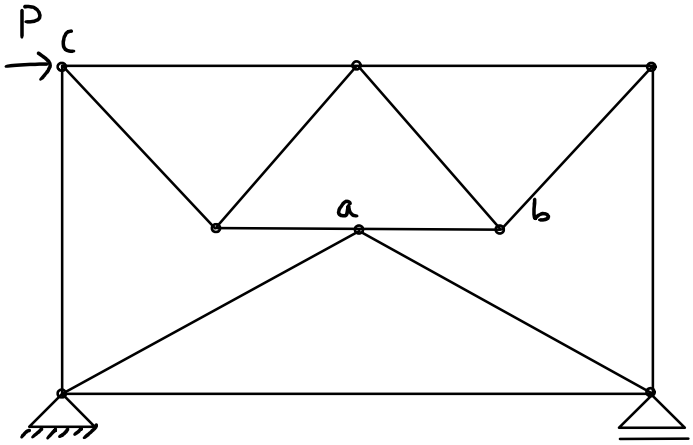


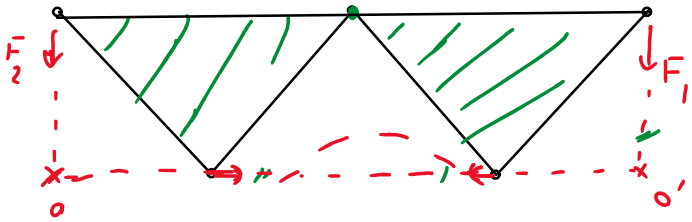
$EA\theta_{ab} = \left(\frac{1}{8}\right)(-P)(4)\times 2 + \left(\frac{3}{8}\right)(-P)(4)\times 2 + 0 + 0 + \ldots$

$$EA\theta_{ab} = \left(\tfrac{1}{8}\right)\left(-\tfrac{P}{2}\right)(4)\times 2 + \left(\tfrac{3}{8}\right)\left(-\tfrac{P}{2}\right)(4)\times 2 + 0 + 0 + \left(-\tfrac{1}{8}\sqrt{2}\right)\left(\tfrac{P}{2}\sqrt{2}\right)(2\sqrt{2}) + \left(-\tfrac{3}{8}\sqrt{2}\right)\left(\tfrac{P}{2}\sqrt{2}\right)(2\sqrt{2})$$

$$+\left(-\tfrac{1}{4}\right)(P)(2)\times 2 = (-3-\sqrt{2})P$$

$$\theta_{ab} = \frac{(3+\sqrt{2})P}{EA}\,\text{CW}$$





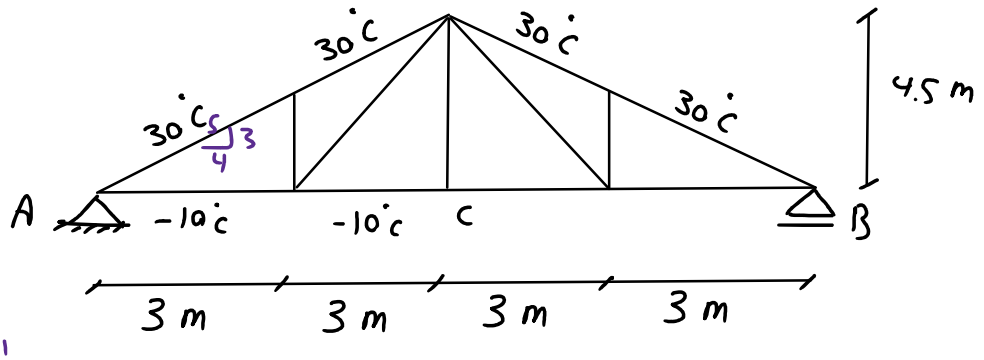
$\sum M_c = 0 \rightarrow F_1$

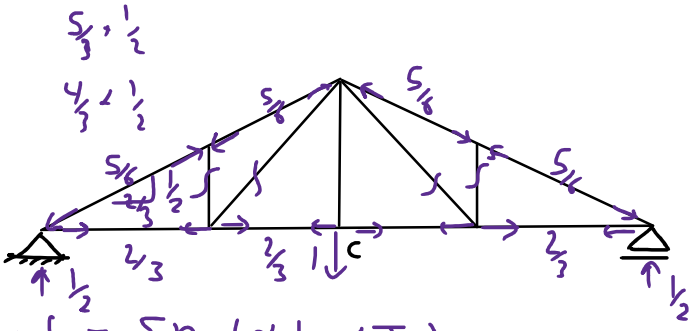
$\sum M_{o'} = 0 \rightarrow F_2$

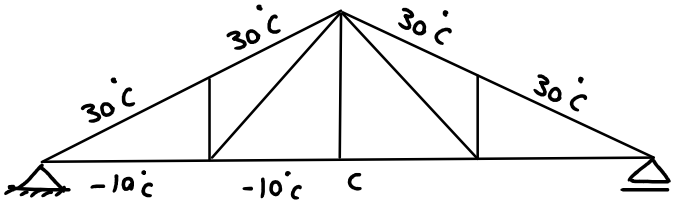
$\delta_{cx} = ?$ $\theta_{ab} = ?$

	n_i	N_i	l_i	$n_i N_i L_i$
ab				
bc				
Σ				

مثال: تغییرمکان قائم گره C را در اثر تغییردماهای نشان داده شده به دست آورید.





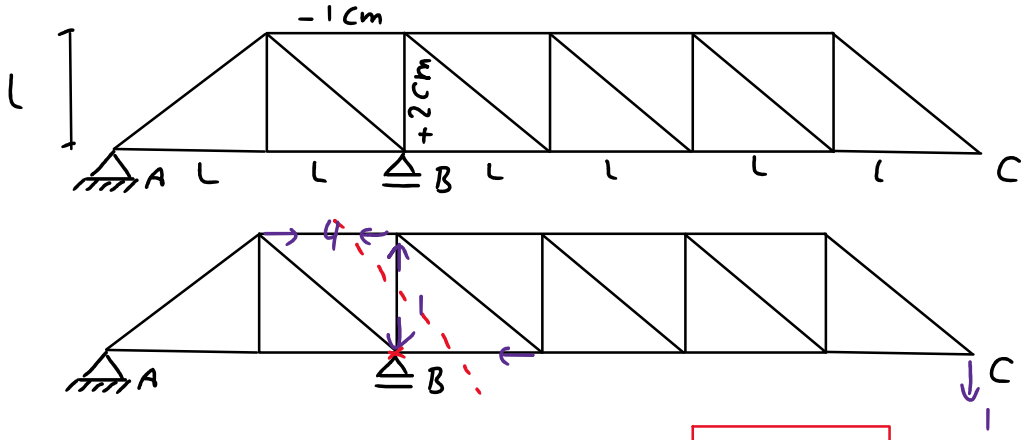


$$1 \times \delta_C = \sum n_i (\alpha_i L_i \Delta T_i)$$

$$1 \times \delta_C = \left(-\frac{5}{6}\right)\left(12\times10^{-6}\times3.75\times30\right)\times4 + \left(\frac{2}{3}\right)\left(12\times10^{-6}\times3\times(-10)\right)\times2 = -0.005\ m$$

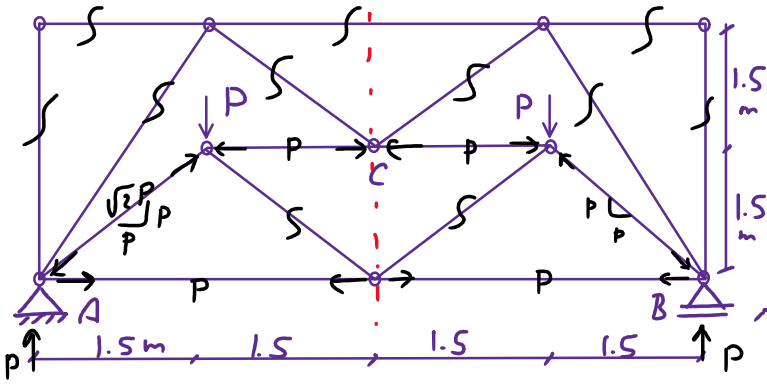
$$\boxed{\delta_C = 5\ mm\uparrow}$$

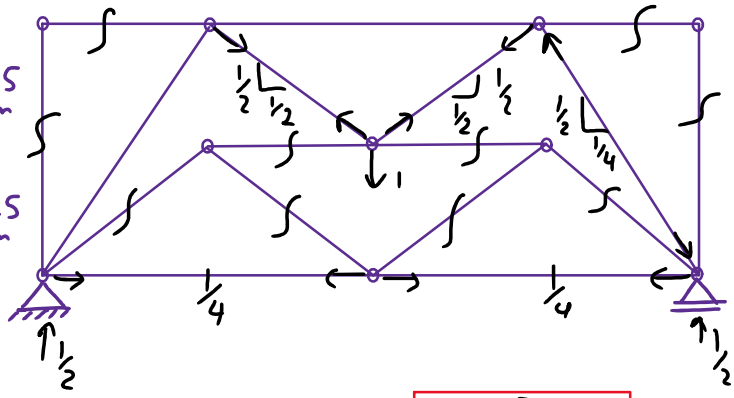
مثال: تغییرمکان قائم گره C در اثر کوتاه و بلند بودن اعضا مطابق شکل چقدر است؟



$$1 \times \delta_C = (-1)(0.02) + (4)(-0.01) = -0.06 \rightarrow \boxed{\delta_C = 6\ cm\uparrow}$$

مثال: جابجایی قائم گره C را تحت بار وارده به دست آورید.

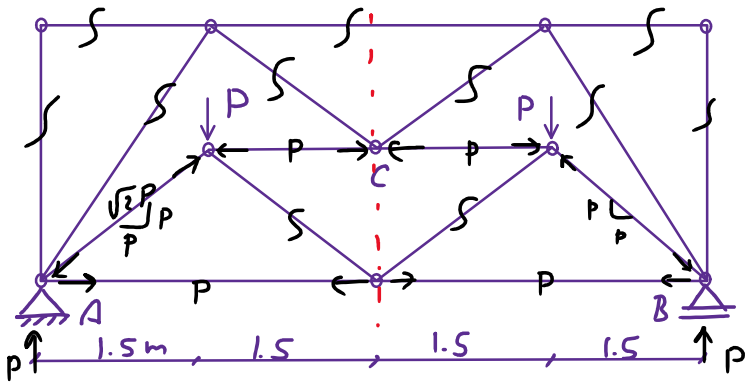


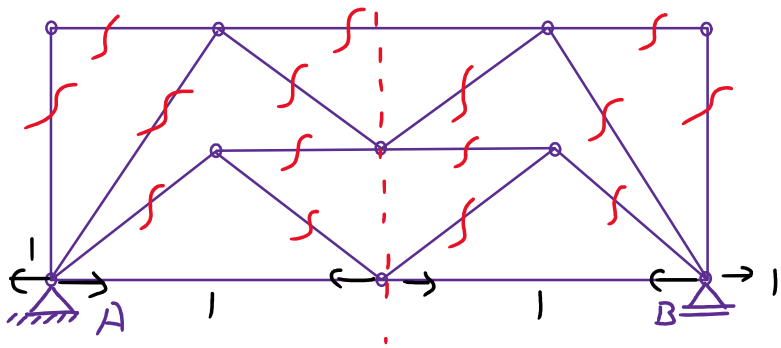


$$1 \times \delta_c = \sum \frac{n_i N_i L_i}{EA_i} = \left(\frac{1}{4}\right)(P)\left(\frac{3}{EA}\right) \times 2 = \frac{3}{2}\frac{P}{EA}$$

$$\boxed{\delta_c = \frac{3}{2}\frac{P}{EA} \downarrow}$$

مثال: اگر در مثال فوق علاوه بر بارهای وارده، کلیه اعضا به مقدار 30°c افزایش دما داشته باشند، جابجایی افق تکیه‌گاه B را به دست آورید.

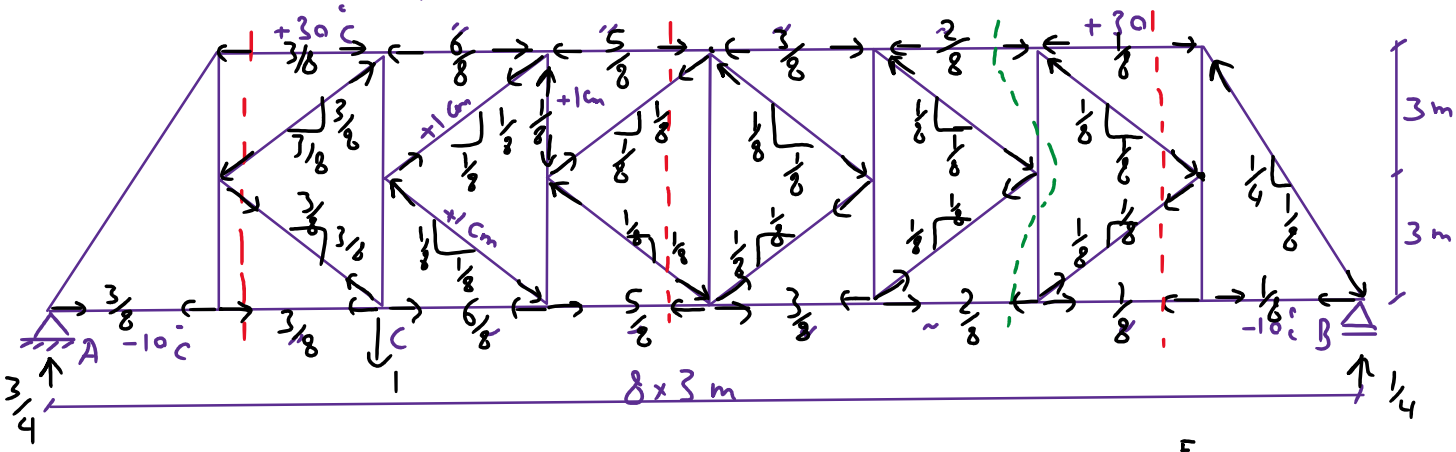




$$1 \times \delta_{1B} = \sum \frac{n_i N_i l_i}{EA} + \sum n_i(\alpha l_i \Delta T_i) = \frac{(1)(P)(3)}{EA} \times 2 + (1)(\alpha(3)(30)) \times 2$$

$$\boxed{\delta_{1B} = \frac{6P}{EA} + 180\alpha \rightarrow}$$

مثال: جابجایی قائم گره C تحت تغییر دما در تار بالای خرپا 30+ و تار پایین خرپا 10- و نقص عضو را محاسب کنید.



$$1 \times \delta = \sum n_i(\alpha L \Delta T) + \left(-\frac{\sqrt{2}}{2}\right)(0.01) + \left(\frac{\sqrt{2}}{2}\right)(0.01) + \left(-\frac{1}{2}\right)(0.01)$$

4

$$1 \times \delta_c = \sum n_i (\alpha L \Delta T) + \left(-\frac{\sqrt{2}}{8}\right)(0.01) + \left(\frac{\sqrt{2}}{8}\right)(0.01) + \left(-\frac{1}{8}\right)(0.01)$$

$$1 \times \delta_c = \alpha \times 3 \times 30 \sum n_i + \alpha \times 3 \times (-10) \sum n_i + \left(-\frac{1}{8}\right)(0.01)$$

$20 - \frac{24}{8} = 3$ $\quad 24 + \frac{28}{8}$

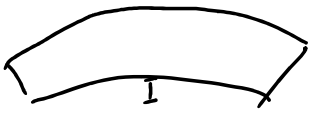
$F_1 \cos\theta = F_2 \cos\theta$ $\quad \frac{3}{4} \cdot \frac{1}{2}$ $\quad \frac{1}{4}$

$\frac{1}{4} \times 6 = F \times 8$

$$1 \times \delta_c = -375\alpha - \frac{0.01}{8} = -\frac{315}{375} \times 12 \times 10^{-6} - \frac{0.01}{8} = -4.5\,mm - 1.25\,mm = -5.75\,mm$$

-3.78 $\quad -5.03$

$$\delta_c = 5.03\,mm \uparrow$$



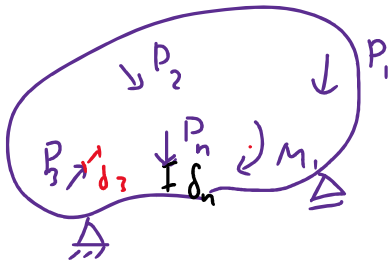
Deflection Energy12

Monday, December 18, 2023 15:53

روش کاستیلیانو

این روش در سال ۱۸۷۹ میلادی توسط آلبرتو کاستیلیانو ارائه شد. در این روش فرض می‌شود در نشست تکیه‌گاه، تغییر دما و نقص عضو نداریم.

در سازه شکل زیر فرض کنیم که تغییر مکان δ_n به اندازه $d\delta_n$ افزایش یابد. در این صورت داریم:



$$U = U(P_1, P_2 \dots P_n, M_1, M_2 \dots M_n, \delta_1, \delta_2 \dots \delta_n, \theta_1, \theta_2 \dots \theta_n)$$

$$\begin{cases} dU = \frac{\partial U}{\partial \delta_n} d\delta_n \\ dU = P_n d\delta_n \end{cases}$$

$$\boxed{\frac{\partial U}{\partial \delta_n} = P_n}$$

قضیه اول کاستیلیانو

$f(x,y,z)$

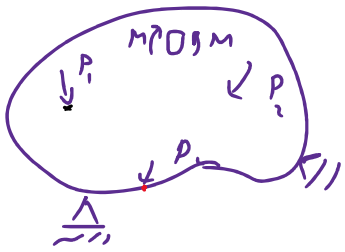
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

اگر بار P_n به مقدار dP_n افزایش یابد، تغییر مکان‌ها به میزان $d\delta_1$ تا $d\delta_n$ تغییر می‌کنند. با توجه به اینکه رفتار سازه خطی است، می‌توان فرض کرد ابتدا بار dP_n و سپس بارهای P_1 تا P_n به سازه اعمال می‌شود. بنابراین داریم:

$$\begin{cases} dU = \frac{\partial U}{\partial P_n} dP_n \\ dU = dP_n \delta_n + \frac{1}{2} dP_n d\delta_n \end{cases}$$

$$\boxed{\frac{\partial U}{\partial P_n} = \delta_n}$$

قضیه دوم کاستیلیانو



$$U = \int \frac{M^2}{2EI} dx \qquad M = P_1 m_1 + P_2 m_2 + \dots P_n m_n$$

$$\frac{\partial U}{\partial P_n} = \int \frac{M\left(\frac{\partial M}{\partial P_n}\right)}{EI} dx = \int \frac{M m_n}{EI} dx \overset{\text{بار واحد}}{=} \delta_n$$

مثال: تغییر مکان نقطه سر آزاد تیر را محاسبه کنید.

$$\delta = \frac{dU}{dP} = \int \frac{M}{EI}\left(\frac{\partial M}{\partial P}\right) dx = \frac{1}{EI}\int_0^L (-Px)(-x)dx = \frac{Px^3}{3EI}\Big|_0^L = \frac{PL^3}{3EI}$$

$M = -Px$

$\frac{\partial M}{\partial P} = -x$

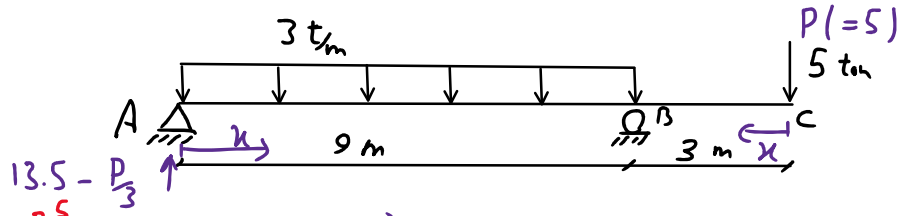
$\frac{1}{2}\int \frac{M^2}{EI}$

$M = -Px - \bar{M}$

$\frac{\partial M}{\partial \bar{M}} = -1$

$$\theta = \frac{\partial U}{\partial \bar{M}} = \int \frac{M}{EI}\left(\frac{\partial M}{\partial \bar{M}}\right)dx = \frac{1}{EI}\int_0^L (-Px - \bar{M})(-1)\,dx = \frac{Px^2}{2EI}\Big|_0^L = \frac{PL^2}{2EI}$$

مثال: تغیر مکان قائم C را به دست آورید. $(EI = 18000\ t.m^2)$



$$\begin{cases} AB: & M = (13.5 - \frac{P}{3})x - \frac{3}{2}x^2 \qquad \frac{\partial M}{\partial P} = -\frac{x}{3} \\ CB: & M = -Px \qquad \frac{\partial M}{\partial P} = -x \end{cases}$$

$$\delta_C = \frac{\partial U}{\partial P} = \int \frac{M}{EI}\left(\frac{\partial M}{\partial P}\right)dx = \frac{1}{EI}\int_0^9 (11.833x - 1.5x^2)\left(-\frac{x}{3}\right)dx + \frac{1}{EI}\int_0^3 (-5x)(-x)\,dx$$

$$\frac{1}{EI}\left(\frac{-11.833}{9}x^3 + \frac{1}{2}\frac{x^4}{4}\right)\Big|_0^9 + \frac{1}{EI}\left(\frac{5x^3}{3}\right)\Big|_0^3 = -\frac{138.375}{EI} + \frac{45}{EI} = \frac{-93.375}{EI}\uparrow$$

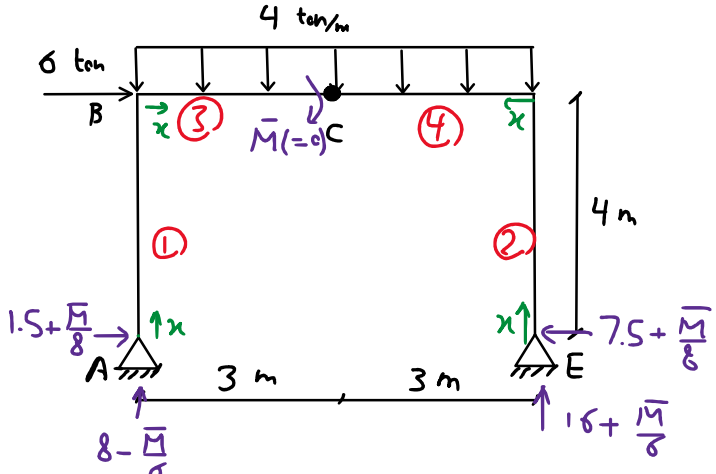
مثال: چرخش سمت چپ مفصل C را به دست آورید.

$$\theta_{Cl} = \frac{\partial U}{\partial \bar{M}} = \int \frac{M}{EI}\frac{\partial M}{\partial \bar{M}}\,dx$$

$\curvearrowleft \sum M_C = 0$

$(6 - \frac{\bar{M}}{6})(3) - 4R_x - 12\times 1.5 + \bar{M} = 0$

$6 + \frac{\bar{M}}{2} = 4R_x$



① $M = -(1.5 + \frac{\bar{M}}{8})x$

② $M = -(7.5 + \frac{\bar{M}}{8})x$

$\frac{\partial M}{\partial \bar{M}} = -\frac{x}{8}$

$\frac{\partial M}{\partial \bar{M}} = -\frac{x}{8}$

$\frac{\partial M}{\partial \bar{M}} = -\frac{1}{8}x$

(2) $M = -(7.5 + \frac{M}{8})x$ $\qquad \frac{\partial M}{\partial M} = -\frac{x}{8}$

(3) $M = (8 - \frac{M}{6})x - (1.5 + \frac{M}{8})(4) - 2x^2$ $\qquad \frac{\partial M}{\partial M} = -\frac{1}{2} - \frac{x}{6}$

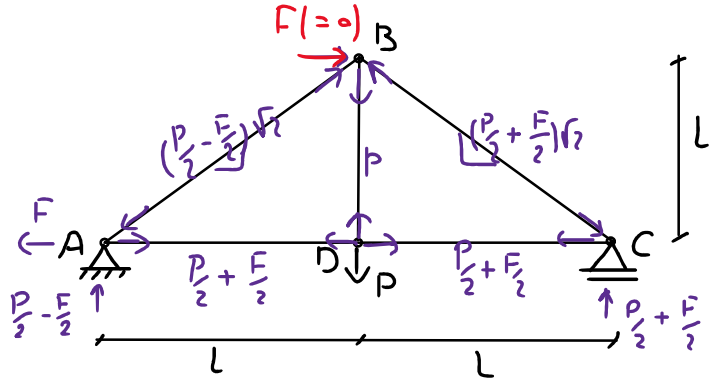
(4) $M = (16 + \frac{M}{6})x - (7.5 + \frac{M}{8})(4) - 2x^2$ $\qquad \frac{\partial M}{\partial M} = -\frac{1}{2} + \frac{x}{6}$

$$EI\theta_{CL} = \int_0^4 (-1.5x)(-\tfrac{x}{8})\,dx + \int_0^4 (-7.5x)(-\tfrac{x}{8})\,dx + \int_0^3 (-2x^2+8x-6)(-\tfrac{1}{2}-\tfrac{x}{6})\,dx$$

$$+ \int_0^3 (-2x^2+16x-30)(-\tfrac{1}{2}+\tfrac{x}{6})\,dx = 4 + 20 - 0.75 + 12.75 \rightarrow \boxed{\theta_{CL} = \frac{3PL^2}{EI}}$$

مثال: جابجایی افقی گره B را به دست آورید.

$$U = \frac{1}{2}\sum \frac{N^2 L}{EA}$$

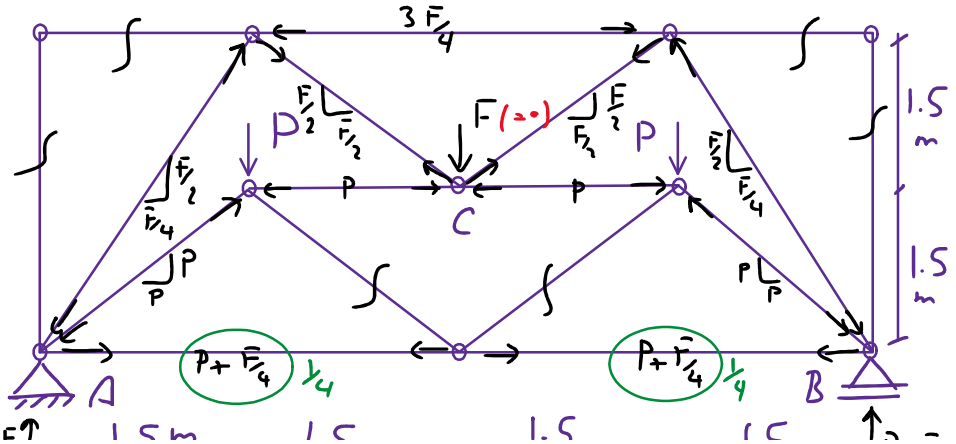


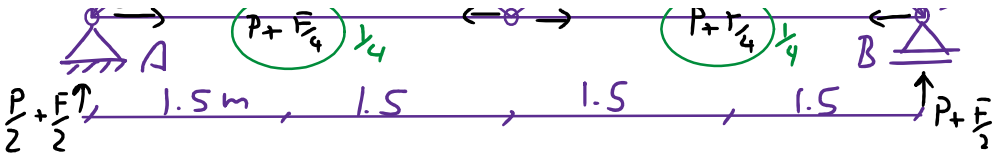
$$\delta_B = \frac{\partial U}{\partial F} = \sum N\left(\frac{\partial N}{\partial F}\right)\frac{L}{EA}$$

$$EA\delta_B = \left(\frac{P}{2} + \frac{F}{2}\right)\left(\frac{1}{2}\right)(L)\times 2 + \sqrt{2}\left(\frac{P}{2} + \frac{F}{2}\right)\left(\frac{\sqrt{2}}{2}\right)(\sqrt{2}L) + \sqrt{2}\left(\frac{P}{2} - \frac{F}{2}\right)\left(\frac{-\sqrt{2}}{2}\right)(\sqrt{2}L) + 0$$

$$\boxed{\delta_B = \frac{PL}{2EA}}$$

مثال: جابجایی قائم گره C را تحت بار وارده به دست آورید.





$$\delta_c = \frac{\partial U}{\partial F} = \sum N\left(\frac{\partial N}{\partial F}\right)\frac{L}{EA} = (P)\left(\frac{1}{4}\right)\frac{3}{EA} \times 2 = \frac{3}{2}\frac{P}{EA}$$

Deflection Energy13

Thursday, December 21, 2023 14:41

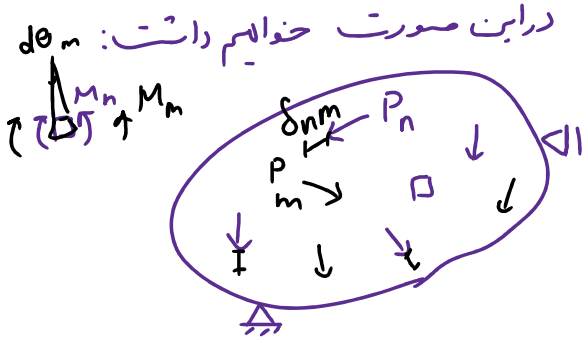
قانون بتی و قانون ماکسول

قانون بتی در سال ۱۸۷۲ میلادی توسط انریکو بتی ارائه شد.

برای به دست آوردن این قضیه، سازه شکل زیر را تحت اثر سیستم نیروهای P_n در نظر بگیرید. حال اگر سیستم نیروهای P_m به سازه وارد شود، تغییر مکان ایجاد شده توسط P_m در نقطه اثر P_n با δ_{nm} نشان داده می‌شود.

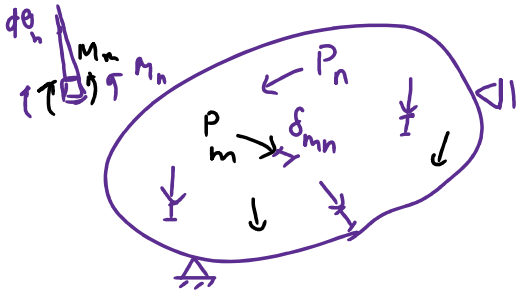
$$W_{ext} = U$$

$$① \quad \sum P_n \delta_{nm} = \int M_n \frac{M_m}{EI} dx$$



حال اگر ابتدا سیستم نیروهای P_m روی سازه موجود شده سپس سیستم نیروهای P_n را به سازه وارد کنیم، داریم:

$$② \quad \sum P_m \delta_{mn} = \int M_m \frac{M_n}{EI} dx$$



بنابراین از ① و ② داریم:

$$\boxed{\sum P_n \delta_{nm} = \sum P_m \delta_{mn}}$$

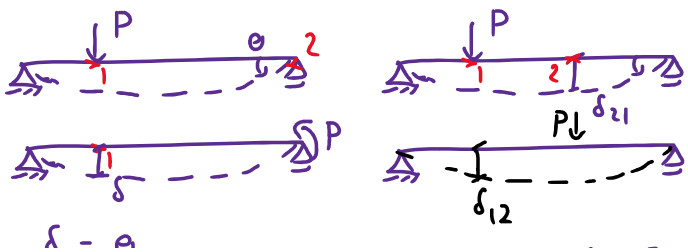
قانون بتی

کار انجام شده توسط سیستم نیروهای P_m به علت تغییر شکل سازه در اثر سیستم نیروهای P_n برابر است با کار انجام شده توسط سیستم نیروهای P_n به علت تغییر شکل سازه در اثر سیستم نیروهای P_m.

قانون ماکسول حالت خاصی از قانون بتی است که در آن تنها یک بار P در دو حالت به سازه وارد می‌شود.

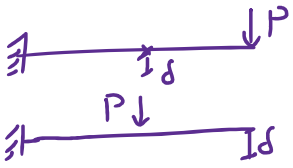
$$\boxed{\delta_{21} = \delta_{12}}$$

قانون ماکسول

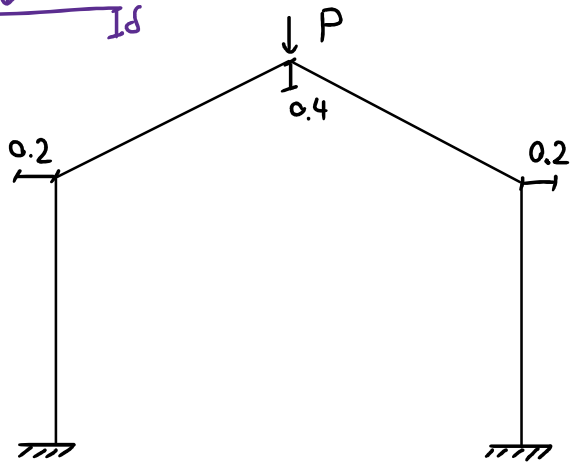


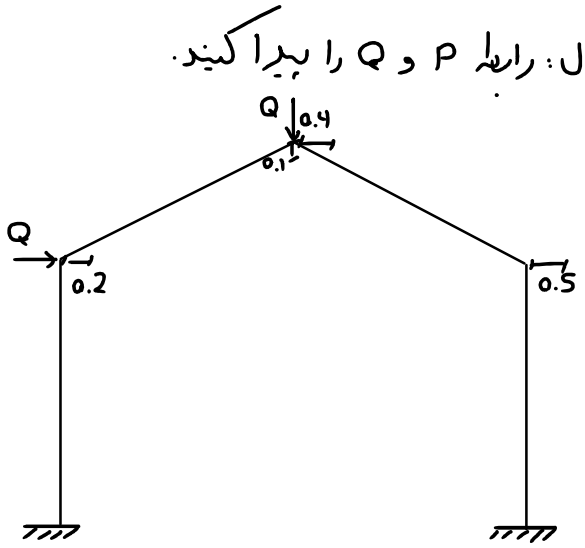
$\delta = \theta$

یعنی تغییرمکان در نقطه ۱ وقتی بار در نقطه ۲ باشد برابر است با تغییرمکان در نقطه ۲ وقتی همان بار در نقطه ۱ اعمال شود.



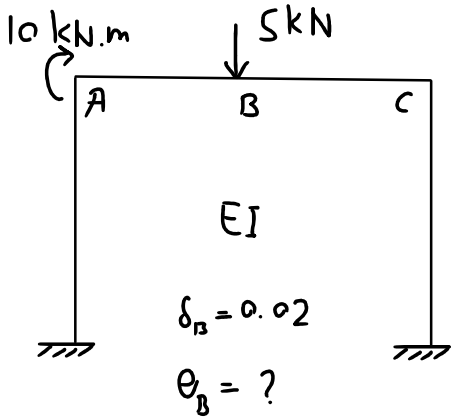
مثال: رابطه P و Q را پیدا کنید.

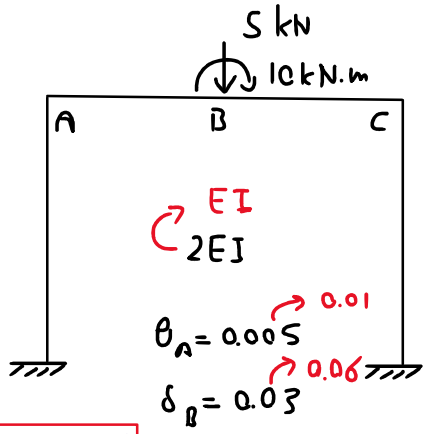




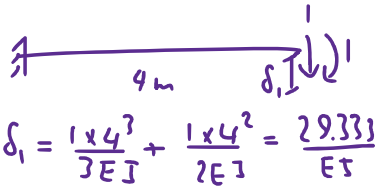
$P(0.1) = -Q(0.2) + Q(0.4) \rightarrow \boxed{P = 2Q}$

مثال:



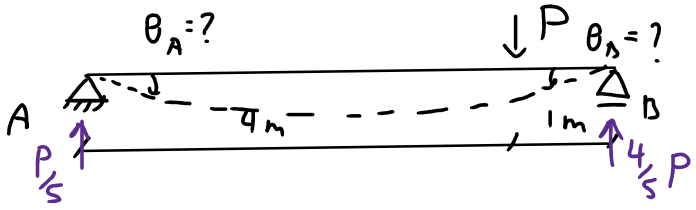


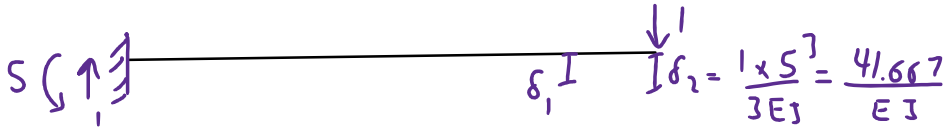
$10 \times 0.01 + 5 \times 0.06 = 5 \times 0.02 + 10\,\theta_B \rightarrow \boxed{\theta_B = 0.02 \text{ rad}}$



$\delta_1 = \frac{1 \times 4^3}{3EI} + \frac{1 \times 4^2}{2EI} = \frac{29.33}{EI}$

مثال:

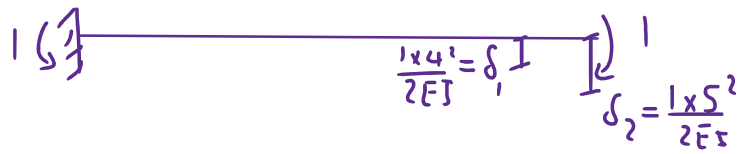




$\delta_2 = \frac{1 \times 5^3}{3EI} = \frac{41.667}{EI}$

$$\left(\frac{P}{5}\right)(0) + P\delta_1 - \left(\frac{4}{5}P\right)\delta_2 = 1\times 0 - 5\theta_A + 1\times 0$$

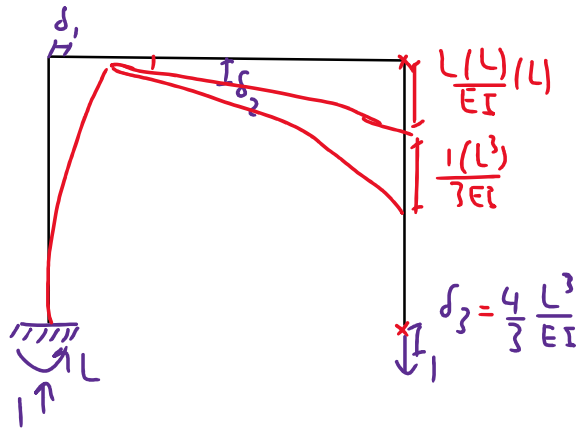
$$P\left(\frac{29.333}{EI}\right) - \left(\frac{4}{5}P\right)\left(\frac{41.667}{EI}\right) = -5\theta_A \rightarrow \boxed{\theta_A = \frac{4}{5}\frac{P}{EI}}$$

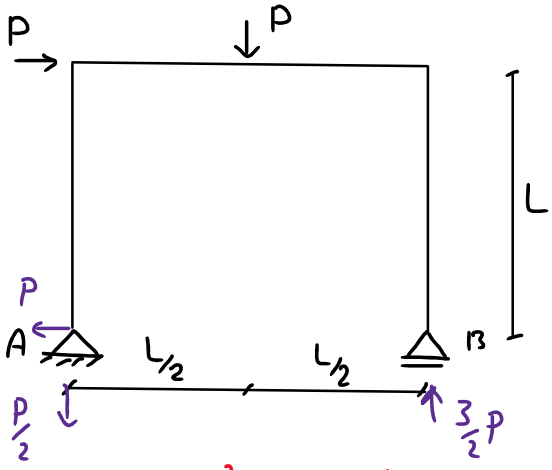


$$P\delta_1 - \left(\frac{4}{5}P\right)\delta_2 = -1\theta_A - 1\theta_B$$

$$P\left(\frac{8}{EI}\right) - \left(\frac{4}{5}P\right)\left(\frac{125}{EI}\right) = -\frac{4}{5}\frac{P}{EI} - \theta_B \rightarrow \boxed{\theta_B = \frac{6}{5}\frac{P}{EI}}$$

مثال: در قاب شکل زیر θ_A را با کمک قانون بتی به دست آورید.

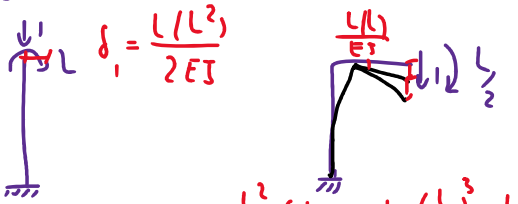




$$L\theta_A = P\delta_1 + P\delta_2 - \frac{3}{2}P\delta_3$$

$$L\theta_A = P\left(\frac{L^3}{2EI}\right) + P\left(\frac{29}{48}\frac{L^3}{EI}\right) - \frac{3}{2}P\left(\frac{4}{3}\frac{L^3}{EI}\right)$$

$$\boxed{\theta_A = \frac{43}{48}\frac{PL^2}{EI}}$$



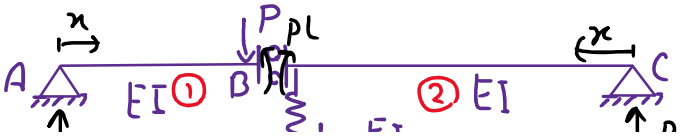
$$\delta_1 = \frac{L^2}{EI}\left(\frac{L}{2}\right) + \frac{1\times\left(\frac{L}{2}\right)^3}{3EI} + \frac{\frac{1}{2}\times\left(\frac{L}{2}\right)^2}{2EI} =$$

$$\frac{1}{2} + \frac{1}{24} + \frac{1}{16} = \frac{24+2+3}{48} = \frac{29}{48}\frac{L^3}{EI}$$

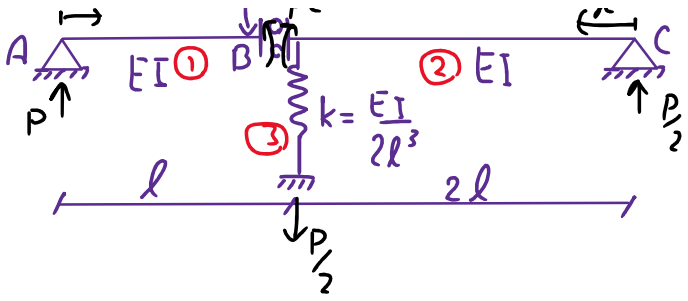
مثال: δ_{BL} و θ_A را با روش کاستیلیانو به دست آورید.

$$\delta_{BL} = \frac{\partial U}{\partial P} \qquad U = \int \frac{M^2}{2EI}dx + \frac{F^2}{2k}$$

$$\delta = \int M\left(\frac{\partial M}{\partial P}\right)dx + F\left(\frac{\partial F}{\partial P}\right)$$



$$\delta_{BL} = \int \frac{M}{EI}\left(\frac{\partial M}{\partial P}\right) dx + \frac{F}{k}\left(\frac{\partial F}{\partial P}\right)$$



① $M = Px$ $\quad \frac{\partial M}{\partial P} = x$

② $M = \frac{P}{2}x$ $\quad \frac{\partial M}{\partial P} = \frac{x}{2}$

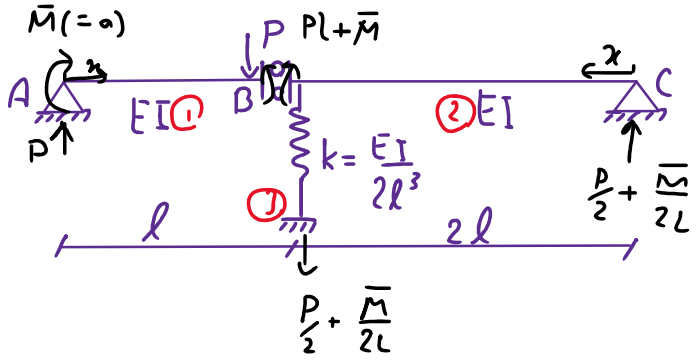
③ $F = \frac{P}{2}$ $\quad \frac{\partial F}{\partial P} = \frac{1}{2}$

$$\delta_{BL} = \frac{1}{EI}\int_0^L (Px)(x)\,dx + \frac{1}{EI}\int_0^{2L}\left(\frac{P}{2}x\right)\left(\frac{x}{2}\right)dx + \frac{P/2}{EI/2\ell^3}\left(\frac{1}{2}\right)$$

$$= \frac{1}{EI}\left(\frac{PL^3}{3} + \frac{P}{4}\frac{(2L)^3}{3}\right) + \frac{PL^3}{2EI} = \frac{3}{2}\frac{PL^3}{EI}$$

$$\boxed{\delta_{BL} = \frac{3}{2}\frac{PL^3}{EI}}$$

$$\theta_A = \frac{\partial U}{\partial \bar{M}} = \int \frac{M}{EI}\frac{\partial M}{\partial \bar{M}}dx + \frac{F}{k}\left(\frac{\partial F}{\partial \bar{M}}\right)$$



① $M = Px + \bar{M}$ $\quad \frac{\partial M}{\partial \bar{M}} = 1$

② $M = \left(\frac{P}{2} + \frac{\bar{M}}{2L}\right)x$ $\quad \frac{\partial M}{\partial \bar{M}} = \frac{x}{2L}$

③ $F = \frac{P}{2} + \frac{\bar{M}}{2L}$ $\quad \frac{\partial F}{\partial \bar{M}} = \frac{1}{2L}$

$$\theta_A = \frac{1}{EI}\int_0^L (Px)(1)\,dx + \frac{1}{EI}\int_0^{2L}\left(\frac{P}{2}x\right)\left(\frac{x}{2L}\right)dx + \frac{P/2}{EI/2\ell^3}\left(\frac{1}{2L}\right)$$

$$= \frac{1}{EI}\left[\frac{PL^2}{2} + \frac{P}{4L}\frac{(2L)^3}{3}\right] + \frac{PL^2}{2EI} = \frac{5}{3}\frac{PL^2}{EI}$$

$$\boxed{\theta_A = \frac{5}{3}\frac{PL^2}{EI}}$$

فصل پنجم :

تحلیل سازه های نامعین به روش نیرویی

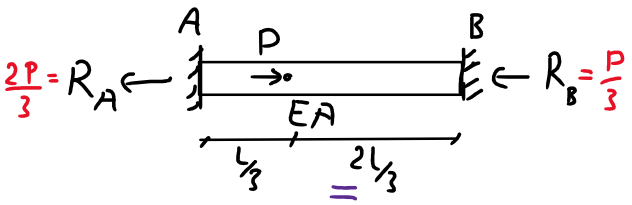
روش تحلیل سازه های نامعین
- ① روش های نیرویی (نرمی) — سازگاری تغییر شکل ها، کاستیگلیانو (تقنیه انرژی حداقل)، تیر مزدوج، سه لنگر
- ② روش های تغییر مکان (سختی) — شیب-افت، توزیع لنگر، کانی

	مجهولات	معادلات
روش نیرویی	نیرو (M, F)	تغییر مکان (θ, δ)
روش تغییر مکان	تغییر مکان (θ, δ)	نیرو (M, F)

روش نیرویی (نرمی)

مجهول R_B

معادله $\delta_B = 0$



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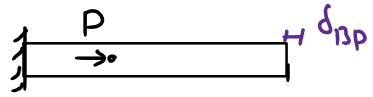
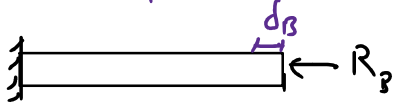
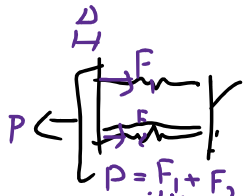


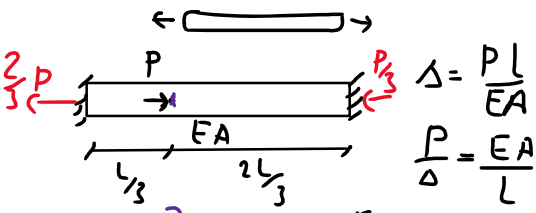
+ سوپرپوزیشن



$$\delta_B = \frac{P(\frac{L}{3})}{EA} - \frac{R_B L}{EA} = 0 \rightarrow \frac{P(\frac{L}{3})}{EA} = \frac{R_B L}{EA} \rightarrow \boxed{R_B = \frac{P}{3}}$$

روش تغییر مکان (سختی)



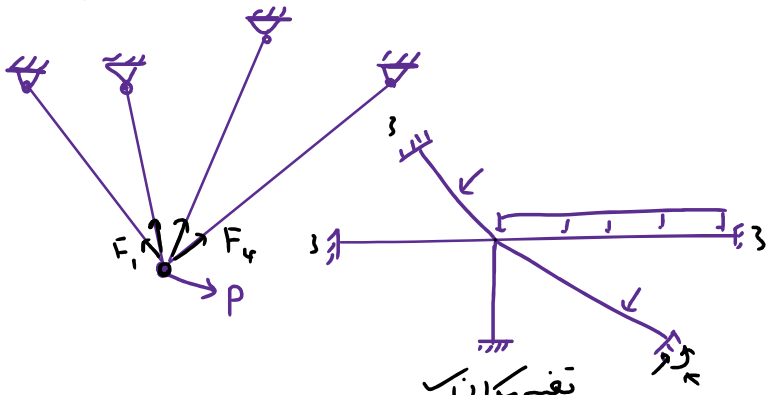


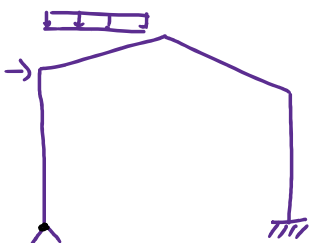
$F_1 + F_2 = P$ معادله نیرویی

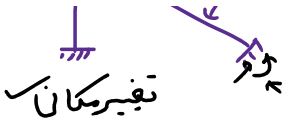
$k_1 \Delta + k_2 \Delta = P$ مجهول Δ

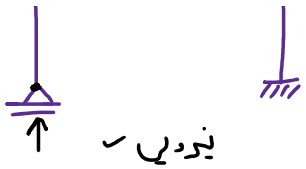
$$\Delta = \frac{P}{k_1 + k_2} = \frac{2}{9}\frac{PL}{EA}$$

$$F_1 = k_1 \Delta = \left(\frac{3EA}{L}\right)\left(\frac{2}{3}\frac{PL}{EA}\right) = \frac{2}{3}P$$



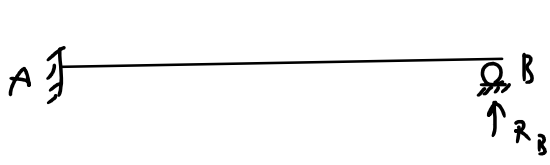


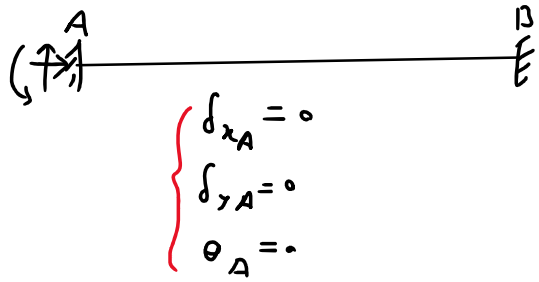


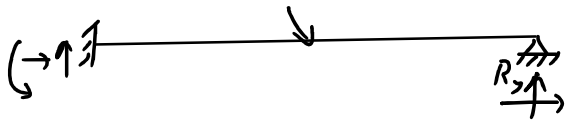


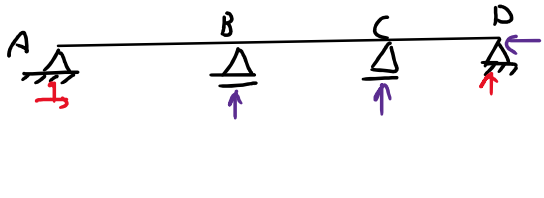
روش سازگاری تغییرشکل ها

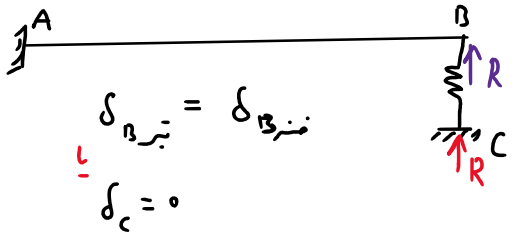
در یک سازه n درجه نامعین، n مجهول نیروی اضافه وجود دارد که با معادلات تعادل به دست نمی آید. اما هر مجهول اضافه، یک معادله تغییرمکان اضافه نیز به همراه دارد. بنابراین با نوشتن n معادله n مجهول کلیه مجهول های نیرویی اضافی به دست می آید. در انتها سایر مجهولات از معادلات تعادل استاتیک تعیین شده و سازه کاملاً تحلیل می شود.





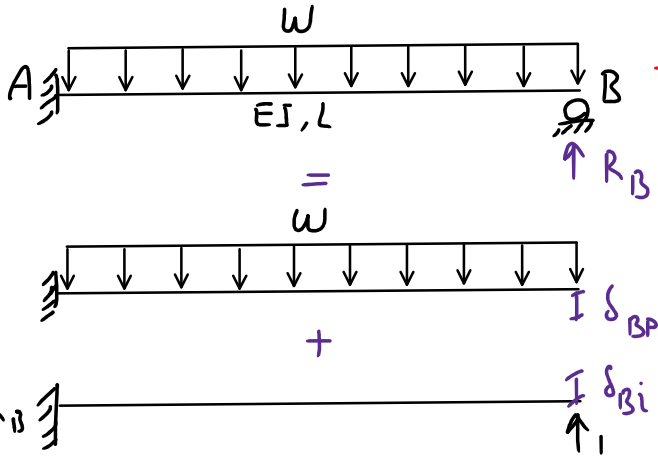






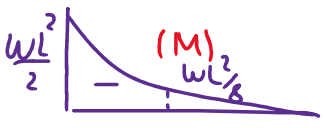
مثال: تیر تکل زیر را تحلیل کنید.

راه حل اول



$\delta_B = 0$

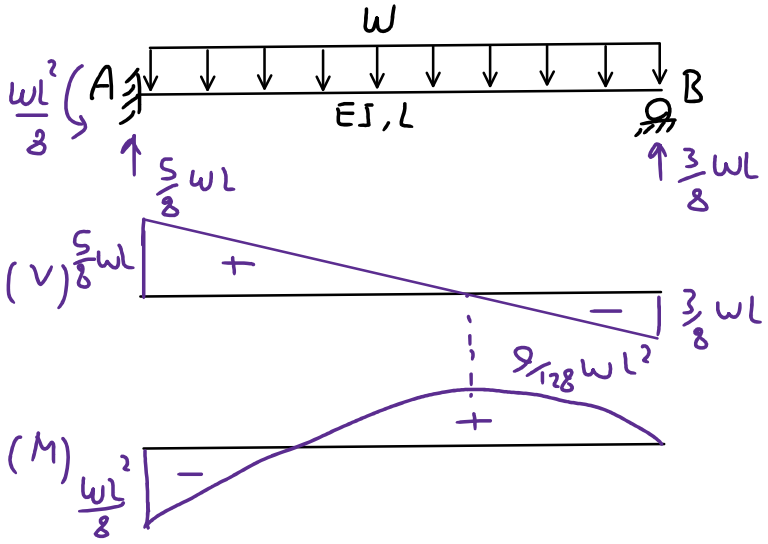
$\delta_{BP} + R_B \delta_{Bi} = 0$

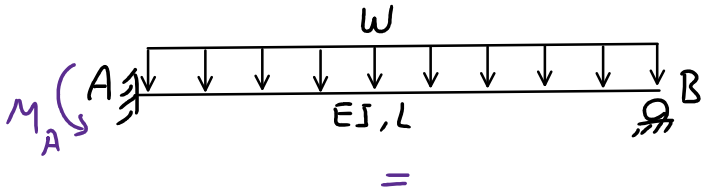


$$\delta_{BP} = \int \frac{m M}{EI} dx = \frac{L}{6}\left[(-\frac{wL^2}{2})(L) + 4(-\frac{wL^2}{8})(\frac{L}{2}) + 0\right] = -\frac{wL^4}{8EI}$$

$$\delta_{Bi} = \int \frac{m^2}{EI}\,dx = \frac{L}{3}(L)(L) = \frac{L^3}{3EI}$$

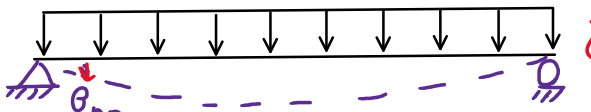
$$-\frac{wL^4}{8EI} + R_B\frac{L^3}{3EI} = 0 \rightarrow \boxed{R_B = \frac{3}{8}wL}$$



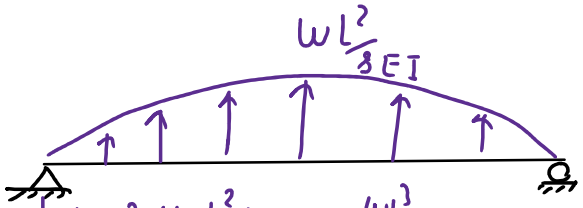


راه حل دوم

=



تیر مزدوج



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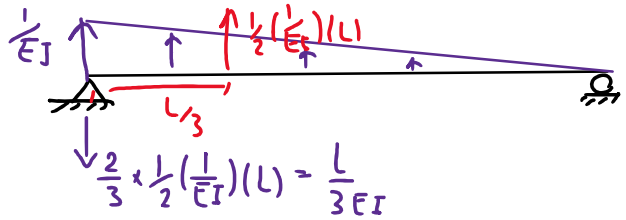
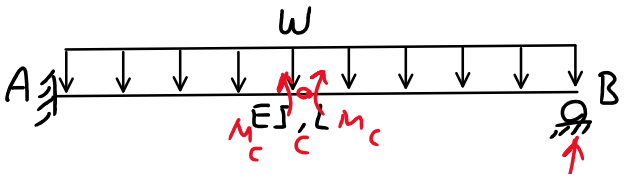
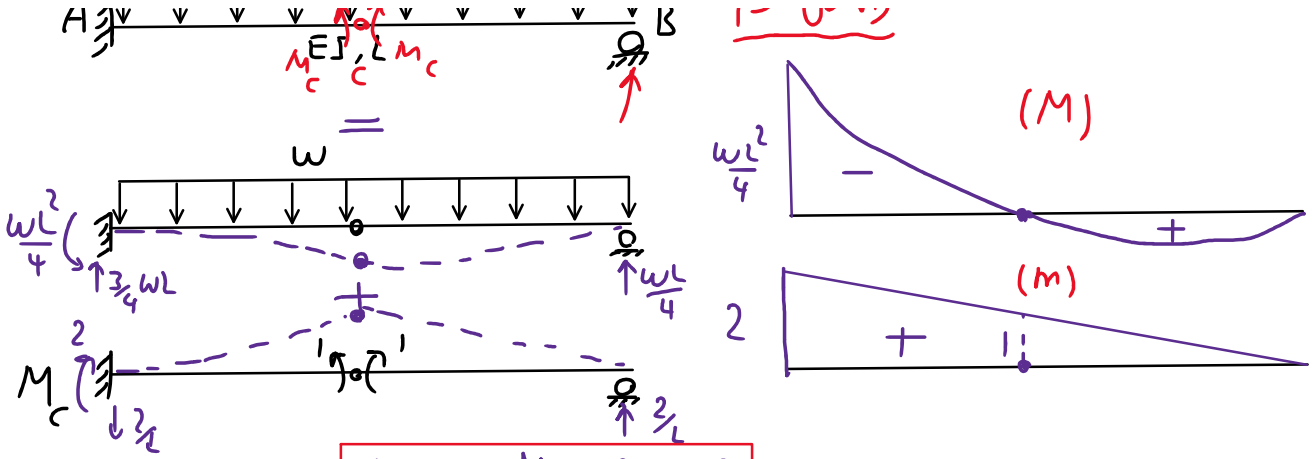


$\theta_A = 0$ $\qquad$ $\boxed{\theta_{AP} + M_A\,\theta_{Ai} = 0}$

$$-\frac{wL^3}{24EI} + M_A\left(\frac{+L}{3EI}\right) = 0 \rightarrow \boxed{M_A = \frac{wL^2}{8}}$$



راه حل سوم

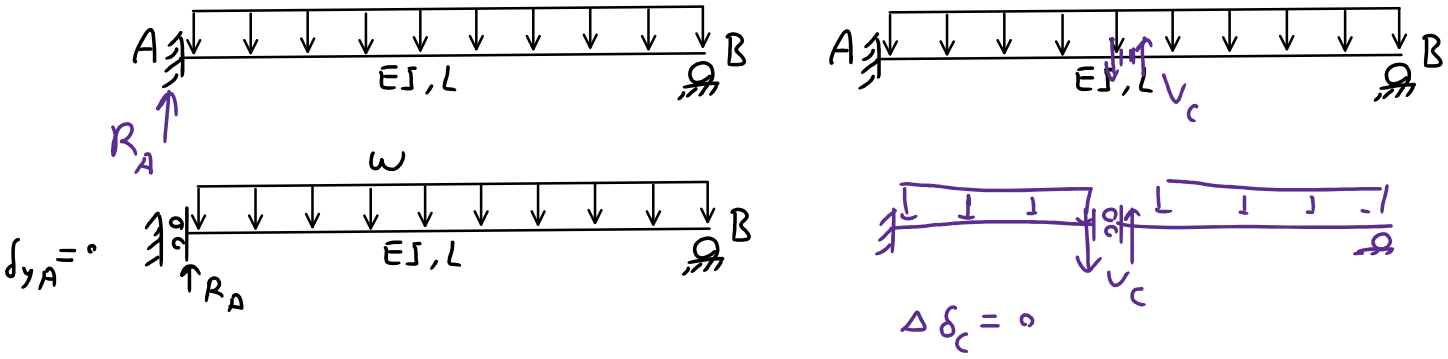


$\Delta\theta_c = 0 \rightarrow \boxed{\Delta\theta_{cp} + M_c \Delta\theta_i = 0}$

$$\Delta\theta_{cp} = \int \frac{mM}{EI}dx = \frac{L}{6}\left[\left(\frac{-wL^2}{4}\right)(2) + 0 + 0\right] = \frac{-wL^3}{12EI}$$

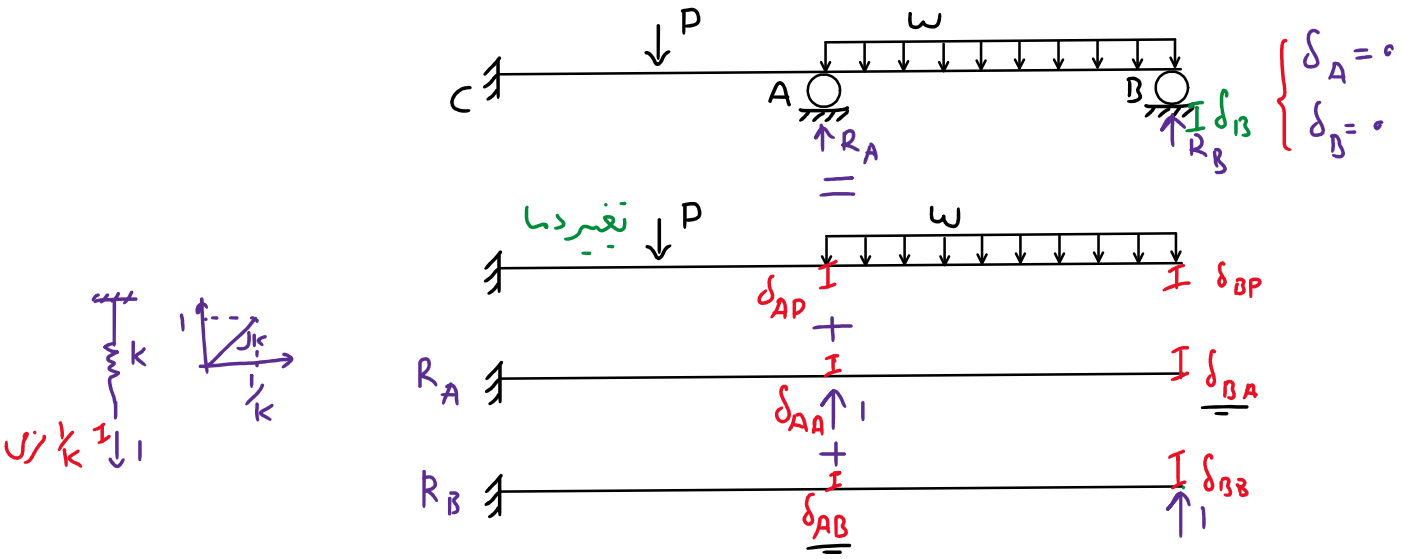
$$\Delta\theta_{ci} = \int \frac{m^2}{EI}dx = \frac{L}{3}(2)(2) = \frac{4}{3}\frac{L}{EI}$$

$$-\frac{wL^3}{12EI} + M_c\left(\frac{4}{3}\frac{L}{EI}\right) = 0 \rightarrow \boxed{M_c = \frac{wL^2}{16}}$$



$\Delta\delta_c = 0$

نوشتن معادلات سازگاری برای سازه ۲ درجه نامعین

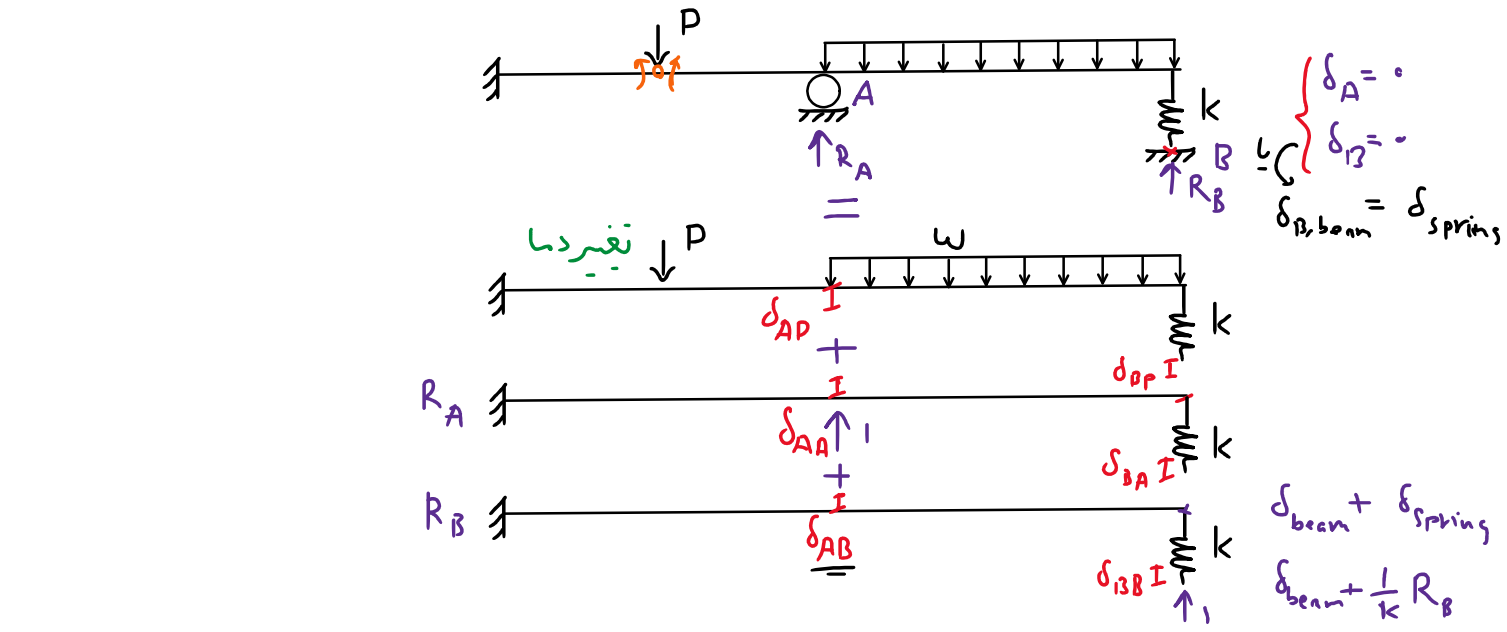


$$\begin{cases} \delta_{AP} + R_A\delta_{AA} + R_B\delta_{AB} = 0 \\ \delta_{BP} + R_A\delta_{BA} + R_B\delta_{BB} = \delta_B \end{cases}$$

$$R_B\delta_{BB} \rightarrow (\delta_{BB} + \frac{1}{k})R_B$$

$$\begin{Bmatrix} \delta_{AP} \\ \delta_{BP} \end{Bmatrix} + \begin{bmatrix} \delta_{AA} & \delta_{AB} \\ \delta_{BA} & \delta_{BB} \end{bmatrix} \begin{Bmatrix} R_A \\ R_B \end{Bmatrix} = \begin{Bmatrix} 0 \\ \delta_B \end{Bmatrix}$$

- $\begin{Bmatrix} \delta_{AP} \\ \delta_{BP} \end{Bmatrix}$: تغییرمکان ناشی از بارخارجی، تغییردما ...
- $\begin{bmatrix} \delta_{AA} & \delta_{AB} \\ \delta_{BA} & \delta_{BB} \end{bmatrix}$: ماتریس نرمی (متقارن) — $\delta_{BB} + \frac{1}{k}$
- $\begin{Bmatrix} R_A \\ R_B \end{Bmatrix}$: مجهولات تکیه‌گاهی
- $\begin{Bmatrix} 0 \\ \delta_B \end{Bmatrix}$



نوشتن معادلات سازگاری برای سازه n درجه نامعین

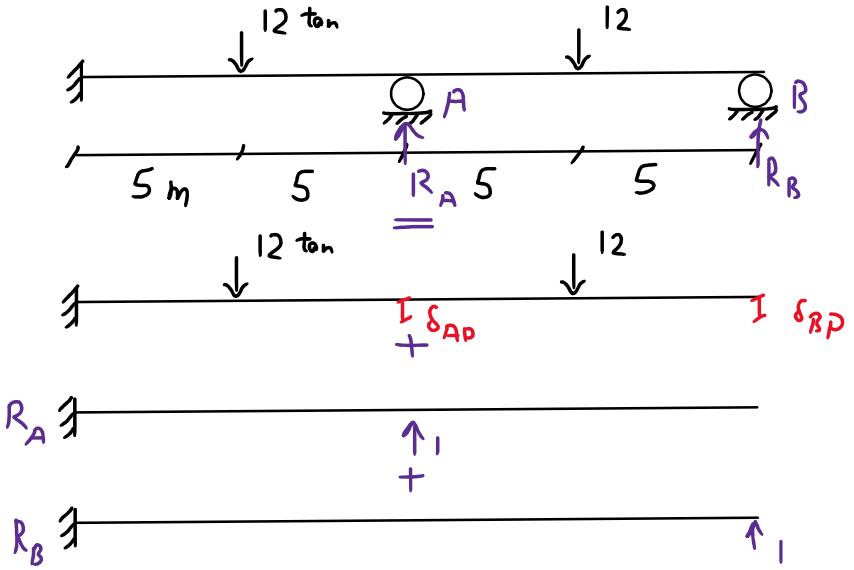
با طور کلی برای یک سازه n درجه نامعین، ابتدا n مجهول اضافه را در سازه مشخص می‌کنیم. سپس معادلات سازگاری را برای هر جهت آنها به صورت زیر می‌نویسیم:

با طوری برای n درجه نامعین، ابتدا n مجهول اضافه را در سازه مشخص می‌کنیم. سپس معادلات سازگاری را براساس جمع آثار قوا به صورت زیر می‌نویسیم:

$$\begin{cases} \delta_{ap} + \delta_{aa} R_a + \delta_{ab} R_b + \delta_{ac} R_c + \cdots + \delta_{an} R_n = 0 \\ \delta_{bp} + \delta_{ba} R_a + \delta_{bb} R_b + \delta_{bc} R_c + \cdots + \delta_{bn} R_n = 0 \\ \vdots \\ \delta_{np} + \delta_{na} R_a + \delta_{nb} R_b + \delta_{nc} R_c + \cdots + \delta_{nn} R_n = 0 \end{cases}$$

$$\begin{Bmatrix} \delta_{ap} \\ \delta_{bp} \\ \vdots \\ \delta_{np} \end{Bmatrix} + \begin{bmatrix} \delta_{aa} & \delta_{ab} & \cdots & \delta_{an} \\ \vdots & & & \\ \delta_{na} & \delta_{nb} & \cdots & \delta_{nn} \end{bmatrix} \begin{Bmatrix} R_a \\ R_b \\ R_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \end{Bmatrix}$$

مثال: تیر شکل زیر را تحلیل کنید.



$\delta_A = 0$

$\delta_B = 0$

$$\delta_{AP} = \left[\frac{12\times5^3}{3EI} + \frac{12\times5^2}{2EI}\times5\right] + \left[\frac{12\times10^3}{3EI} + \frac{60\times10^2}{2EI}\right] = \frac{1250}{EI} + \frac{7000}{EI} = -\frac{8250}{EI}$$

$$\delta_{AA} = \frac{1\times10^3}{3EI} = \frac{333.33}{EI} \qquad \delta_{AB} = \frac{1\times10^3}{3EI} + \frac{10\times10^2}{2EI} = \frac{833.33}{EI}$$

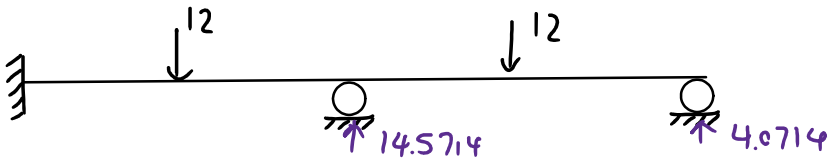
$$\delta_{BP} = \left[\frac{12\times5^3}{3EI} + \frac{12\times5^2}{2EI}\times15\right] + \left[\frac{12\times15^3}{3EI} + \frac{12\times15^2}{2EI}\times5\right] = \frac{2750}{EI} + \frac{20250}{EI} = -\frac{23000}{EI}$$

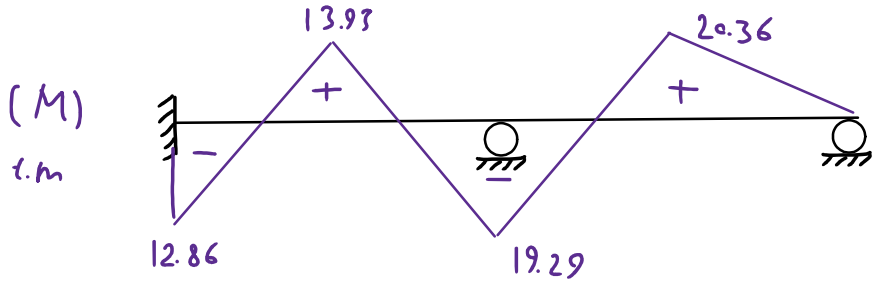
$$\delta_{BA} = \delta_{AB} = \frac{833.33}{EI} \qquad \delta_{BB} = \frac{1\times20^3}{3EI} = \frac{2666.67}{EI}$$

$$\begin{cases} -8250 + 333.33 R_A + 833.33 R_B = 0 \\ -23000 + 833.33 R_A + 2666.67 R_B = 0 \end{cases} \rightarrow$$

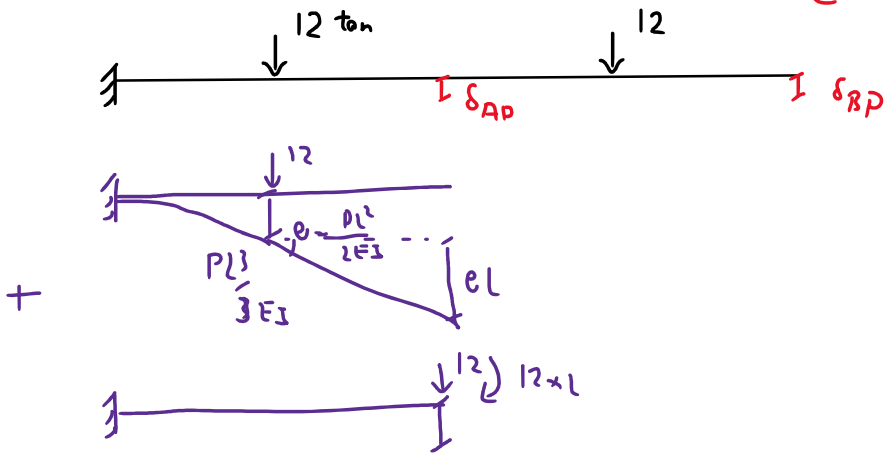
$R_A = 14.5714$ ton

$R_B = 4.0714$ ton

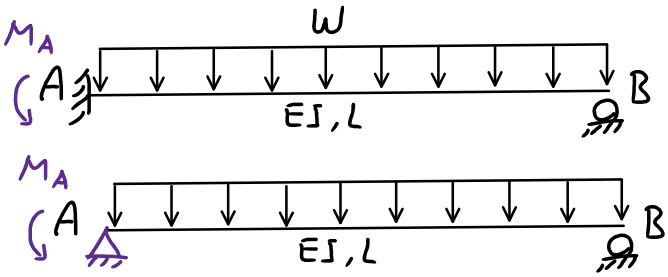


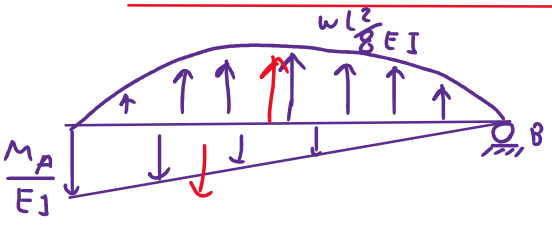


توضیح محاسبه تغییر مکان ها



تحلیل تیر نامعین با معادلات تعادل تیر مزدوج





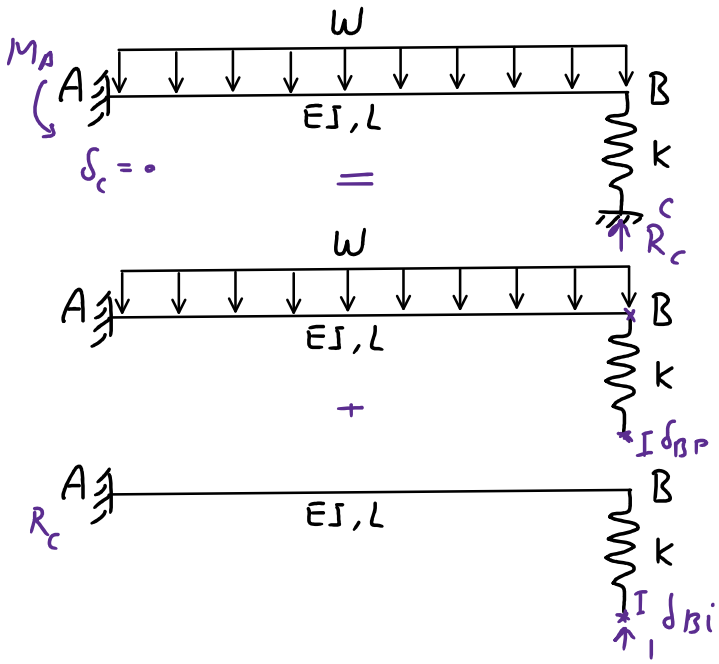
تیر اصلی $\theta_A = 0$ $\longrightarrow$ تیر مزدوج $\Sigma M_B = 0$ $\longrightarrow$

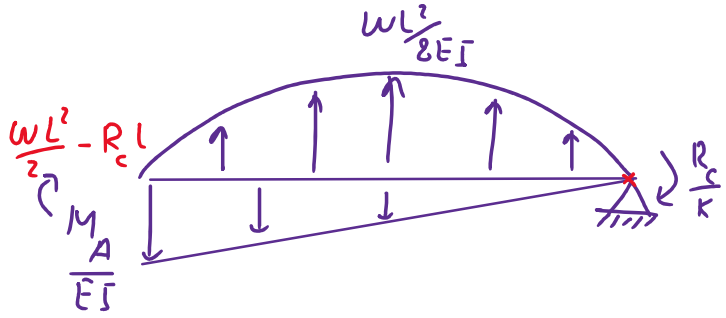
$$\frac{2}{3}\left(\frac{WL^2}{8EI}\right)(L)\left(\frac{L}{2}\right) - \frac{1}{2}\left(\frac{M_A}{EI}\right)(L)\left(\frac{2}{3}L\right) = 0$$

$$\frac{WL^4}{24EI} = \frac{1}{3}\frac{M_A L^2}{EI} \rightarrow \boxed{M_A = \frac{WL^2}{8}}$$

* در واقع به جای نوشتن معادلات سازگاری در تیر اصلی می‌توان معادلات تعادل در تیر مزدوج را نوشت.

مثال: عکس العمل تکیه‌گاه را به روش سازگاری تغییر شکل‌ها و معادلات تعادل تیر مزدوج بدست آورید.





$$\frac{2}{3}\left(\frac{WL^2}{8EI}\right)(L)\left(\frac{L}{2}\right) - \frac{1}{2}\left(\frac{WL^2}{2EI} - \frac{R_c L}{EI}\right)(L)\left(\frac{2}{3}L\right) + \frac{R_c}{k} = 0$$

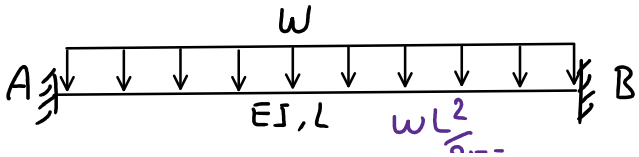
$$\frac{1-4}{24}\frac{WL^4}{EI} = R_c\left(\frac{L^3}{3EI} + \frac{1}{k}\right)$$

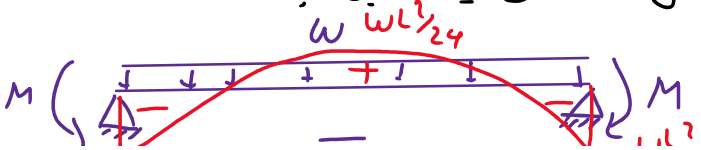
$$\boxed{R_c = \frac{3}{8}WL\frac{1}{\left(1 + \frac{3EI}{kL^3}\right)}}$$

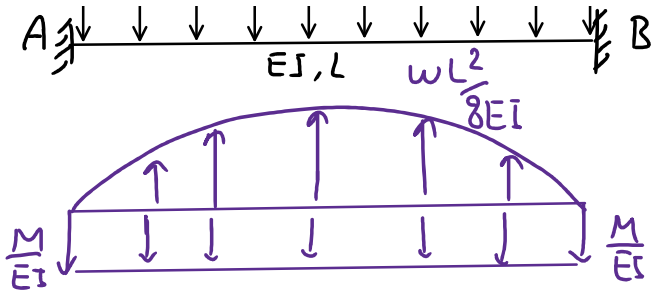
$$\delta_{BP} + R_c\,\delta_{Bi} = 0$$

$$\frac{WL^4}{8EI} - \left(\frac{R_c L^3}{3EI} + \frac{R_c}{k}\right) = 0 \rightarrow \boxed{R_c = \frac{\frac{3}{8}WL}{1 + \frac{3EI}{kL^3}}}$$

مثال: تیر شکل زیر را تحلیل کنید.



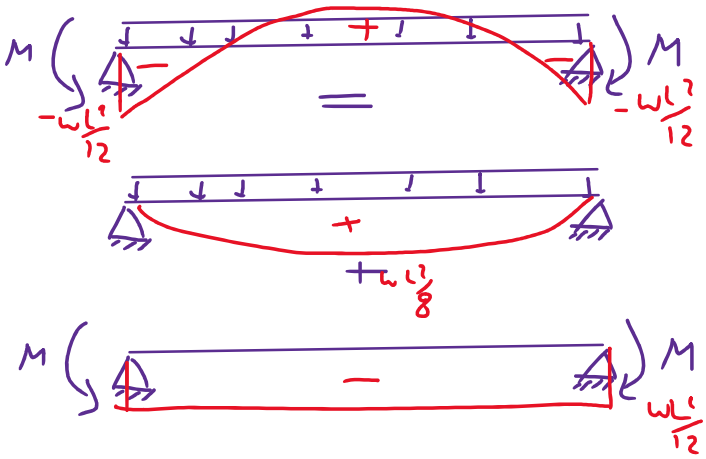




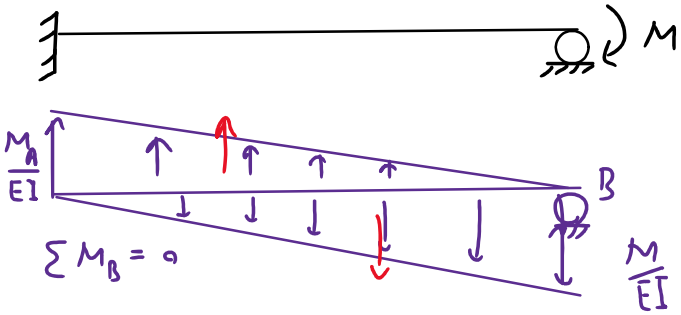
$\sum F_y = 0$

$$\frac{2}{3}\left(\frac{wL}{8EI}\right)(L) - \left(\frac{M}{EI}\right)(L) = 0$$

$$M_A = M_B = \frac{-wL^2}{12}$$

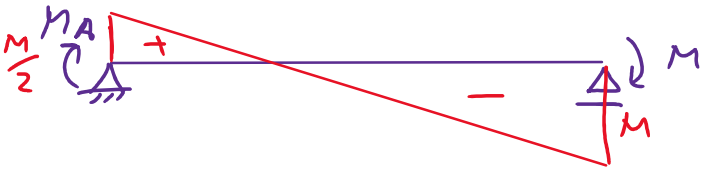


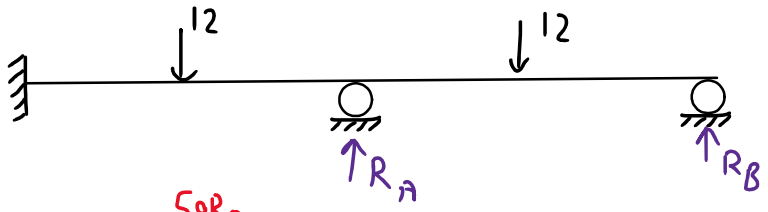
مثال :

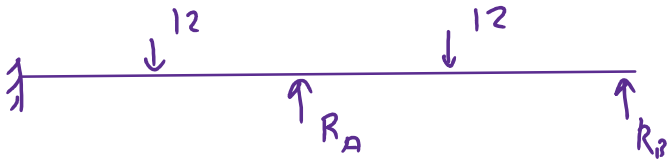


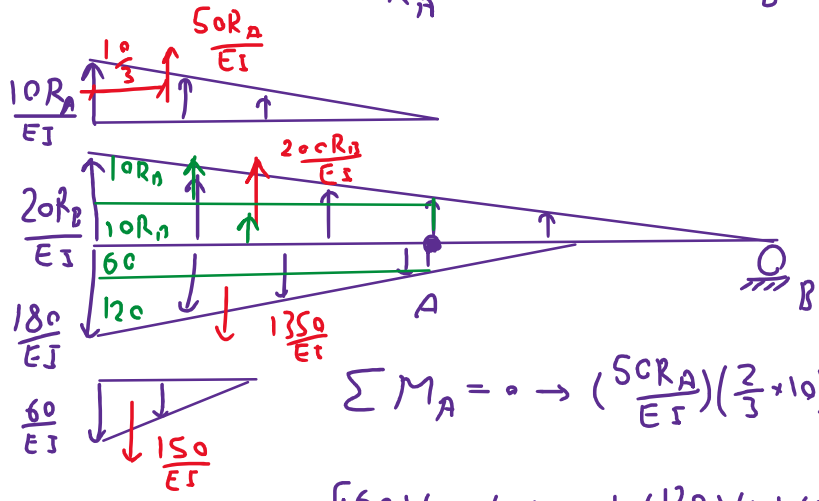
$$\frac{1}{2}\left(\frac{M_A}{EI}\right)(L)\left(\frac{2}{3}L\right) - \frac{1}{2}\left(\frac{M}{EI}\right)(L)\left(\frac{L}{3}\right) = 0$$

$$M_A = \frac{M}{2}$$









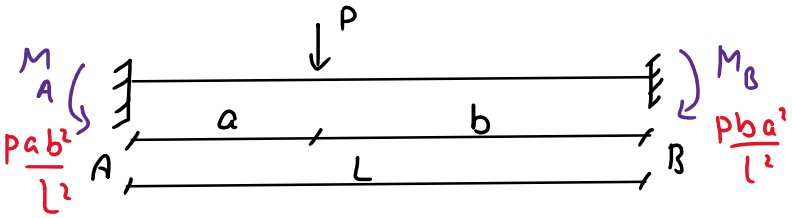
$$\sum M_A = 0 \rightarrow \left(\frac{50R_A}{EI}\right)\left(\frac{2}{3}\times 10\right) + \left(\frac{10R_B}{EI}\right)(10)(5) + \frac{1}{2}\left(\frac{10R_B}{EI}\right)(10)\left(\frac{2}{3}\times 10\right)$$

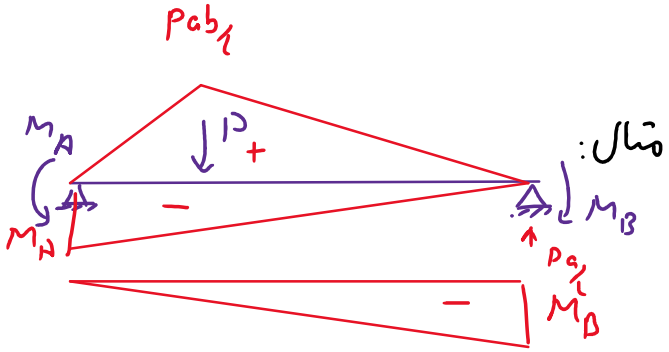
$$-\left[\left(\frac{60}{EI}\right)(10)(5) + \frac{1}{2}\left(\frac{120}{EI}\right)(10)\left(\frac{2}{3}\times 10\right)\right] - \left(\frac{150}{EI}\right)\left(\frac{25}{3}\right) = 0$$

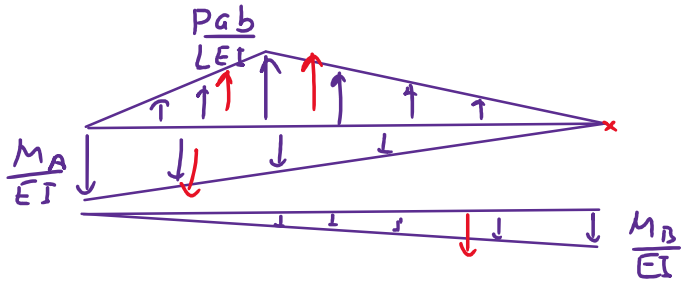
$$* \quad 333.33 R_A + 833.33 R_B - 8250 = 0$$

$$\Sigma M_B = 0 \rightarrow \left(\frac{50 R_A}{EI}\right)(16.67) + \left(\frac{200 R_B}{EI}\right)\left(\frac{2}{3}\times 20\right) - \left(\frac{1350}{EI}\right)(15) - \left(\frac{150}{EI}\right)(18.33) = 0$$

$$* \quad 833.33 R_A + 2666.67 R_B - 27000 = 0$$







$$\Sigma F_y = 0 \rightarrow \frac{1}{2}\left(\frac{Pab}{LEI}\right)(L) - \frac{1}{2}\left(\frac{M_A}{EI}\right)(L) - \frac{1}{2}\left(\frac{M_B}{EI}\right)(L) = 0$$

$$\Sigma M_B = 0 \rightarrow \frac{1}{2}\left(\frac{Pab}{LEI}\right)(b)\left(\frac{2}{3}b\right) + \frac{1}{2}\left(\frac{Pab}{LEI}\right)(a)\left(b+\frac{a}{3}\right) - \left(\frac{M_A L}{2EI}\right)\left(\frac{2L}{3}\right) - \left(\frac{M_B L}{2EI}\right)\left(\frac{1}{3}L\right) = 0$$

$$-\begin{cases} M_A + M_B = \dfrac{Pab}{L} \\ 2M_A + M_B = \dfrac{Pab}{L} + \dfrac{Pab^2}{L^2} \end{cases}$$

$$\boxed{M_A = \frac{Pab^2}{L^2}} \qquad \boxed{M_B = \frac{Pba^2}{L^2}}$$

جربی نویس

$$6\times \frac{Pab}{L}\left(\frac{b^2}{3} + \frac{ab}{2} + \frac{a^2}{6}\right) = \frac{2}{3}\times\frac{L^2}{3}\left(M_A + \frac{M_B}{2}\right)$$

$$\frac{Pab}{L^3}\left(2b^2 + 3ab + a^2\right) = 2\left(M_A + \frac{M_B}{2}\right)$$

$$(a+b)^2 + b^2 + ab = L^2 + bL$$

$$\frac{Pab}{L^2}(L+b) = 2\left(M_A + \frac{M_B}{2}\right)$$

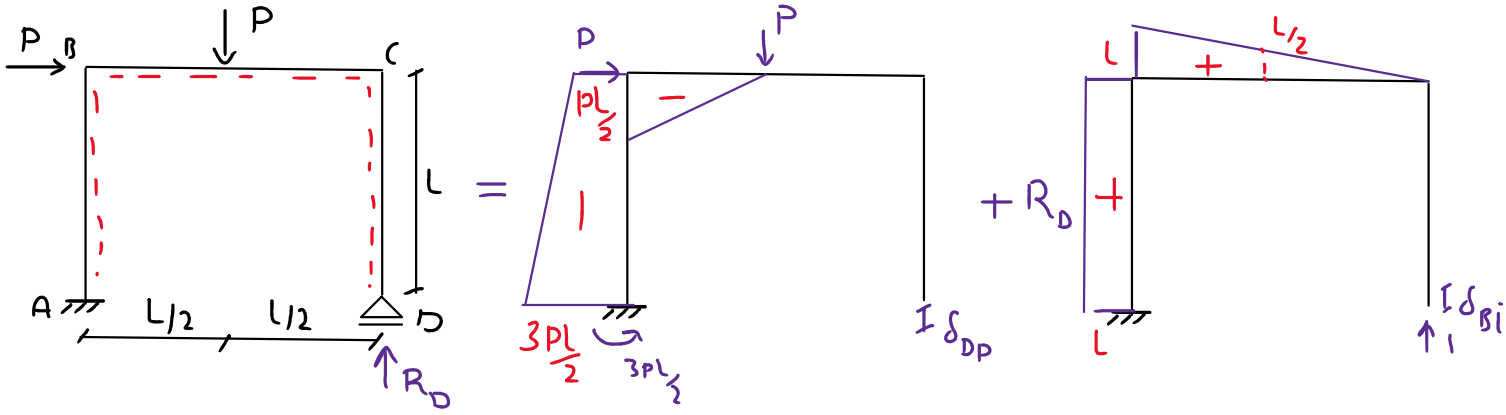
$$\frac{Pab^2}{L^2} \quad \frac{Pab}{L} \quad \frac{Pab}{L}\cdot\frac{L}{L}$$

$$-\frac{Pa^2b^2}{L^2}+\frac{Pab}{L}=\frac{Pab}{L}\left(1-\frac{b}{L}\right)$$

$$\left(\frac{L-b}{L}\right)a$$

تحلیل سازه های نامعین

مثال: عکس العمل تکیه گاه D را به دست آورید.



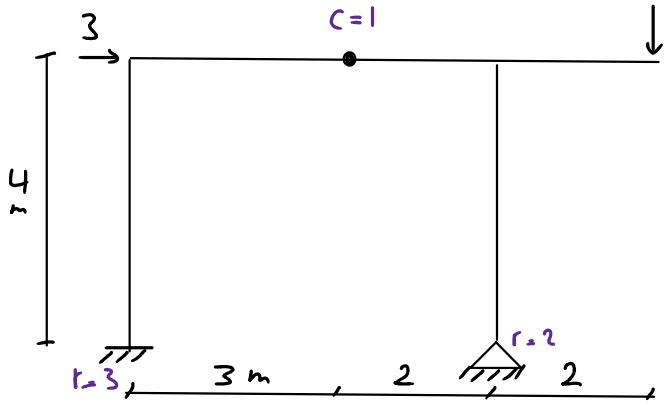
$$\delta_D = 0 \rightarrow \delta_{DP} + R_D \delta_{Di} = 0$$

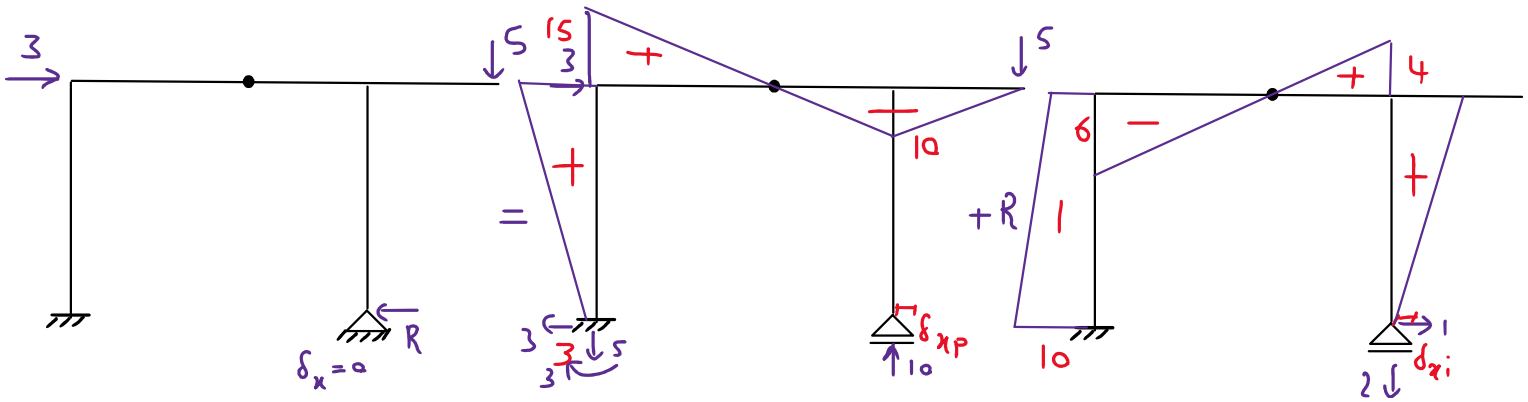
$$1 \times \delta_{DP} = \int \frac{m M}{EI} dx = \left(\frac{L}{6}\right)\left[2\left(\frac{-PL}{2}\right)(L) + \left(\frac{-PL}{2}\right)\left(\frac{L}{2}\right)\right] + (L)(-PL)(L) = -\frac{53}{48}\frac{PL^3}{EI}$$

$$1 \times \delta_{Di} = \int \frac{m^2}{EI} dx = \left(\frac{L}{3}\right)(L^2) + (L^2)(L) = \frac{4}{3}\frac{L^3}{EI}$$

$$-\frac{53}{48}\frac{PL^3}{EI} + R_D\left(\frac{4}{3}\frac{L^3}{EI}\right) = 0 \rightarrow \boxed{R_D = \frac{53}{64}P}$$

مثال: دیاگرام لنگر قاب شکل زیر را رسم نمایید.



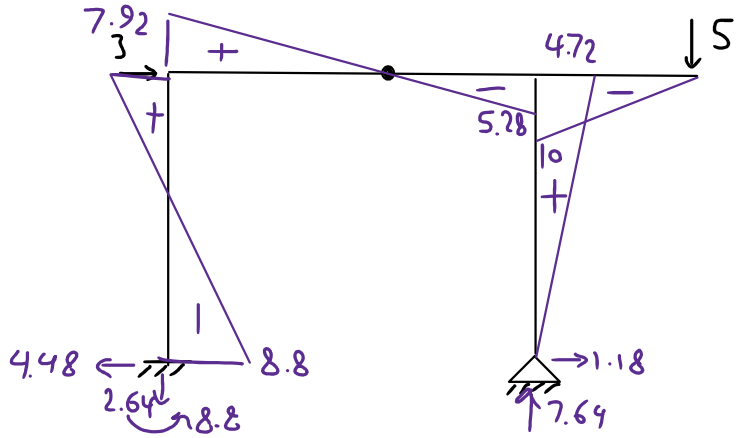


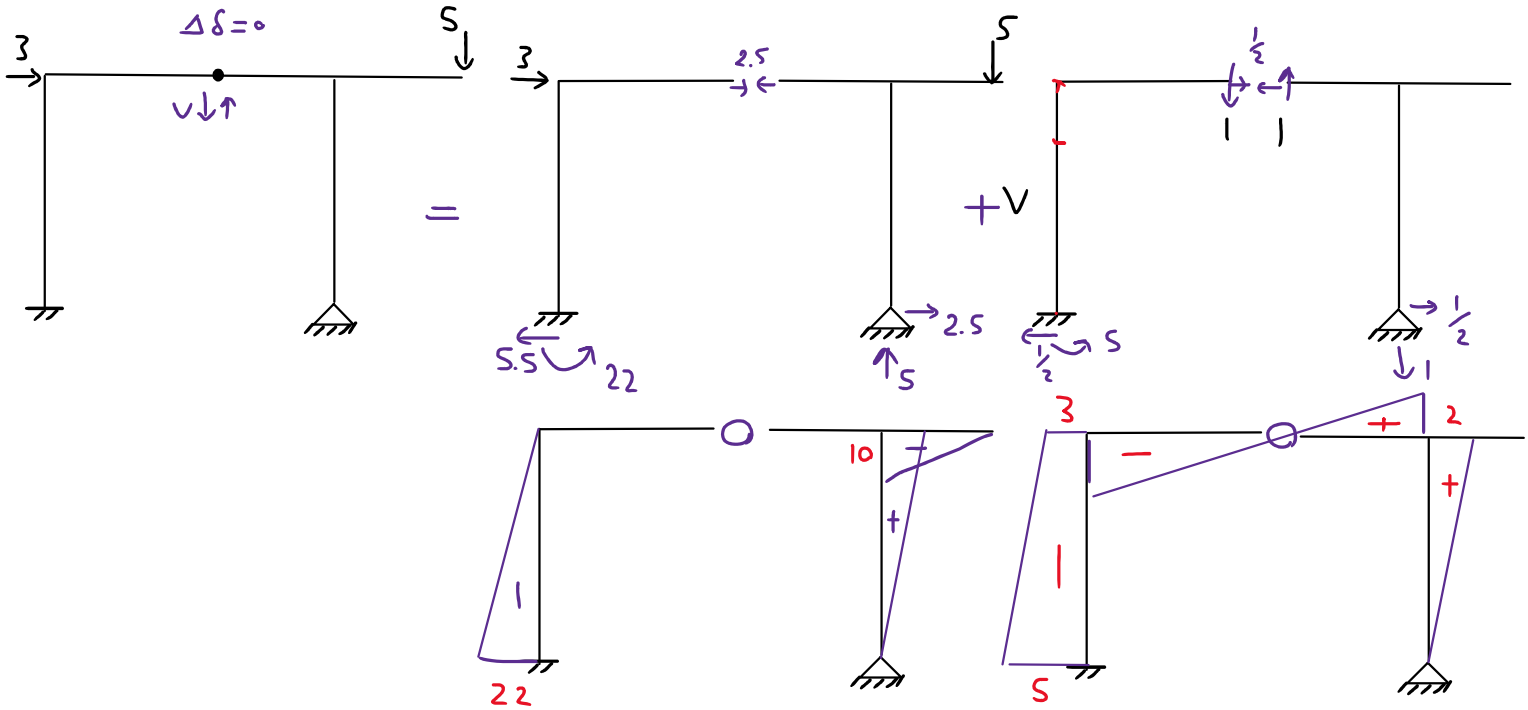
$$1 \times \delta_{xp} = \int \frac{mM}{EI}\,dx = \left(\frac{4}{6}\right)\left[(3)(-10)+4(9)(-8)+(15)(-6)\right]+\left(\frac{3}{3}\right)(15)(-6)+\left(\frac{2}{3}\right)(-10)(4) =$$

$$\delta_{xp} = \frac{-388.67}{EI}$$

$$1 \times \delta_{xi} = \int \frac{m^2}{EI}\,dx = \left(\frac{4}{6}\right)\left[(-10)^2+4(-8)^2+(-6)^2\right]+\left(\frac{3}{3}\right)(-6)^2+\left(\frac{2}{3}\right)(4)^2+\left(\frac{4}{3}\right)(4)^2 = \frac{329.33}{EI}$$

$$\delta_{xp} + R\,\delta_{xi} = 0 \rightarrow \frac{-388.67}{EI} + R\,\frac{329.33}{EI} = 0 \rightarrow \boxed{R = 1.18} \rightarrow$$



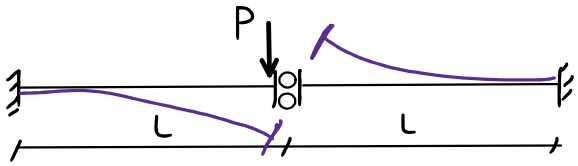


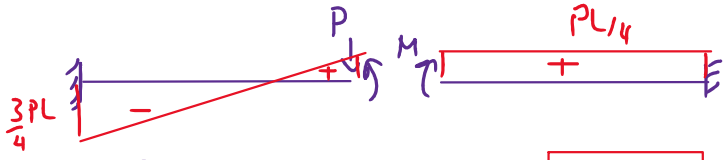
$$1 \times \Delta\delta_p = \int \frac{mM}{EI}\,dx = \left(\frac{4}{3}\right)\left[2(-22)(-5)+(-22)(-3)\right]+\left(\frac{4}{3}\right)(10)(2) = \frac{217.3}{EI}$$

$$1 \times \Delta\delta_i = \int \frac{m^2}{EI}\,dx = \left(\frac{4}{6}\right)\left[(-5)^2 + 4(-4)^2 + (-3)^2\right] + \left(\frac{3}{3}\right)(-3)^2 + \left(\frac{2}{3}\right)(2)^2 + \left(\frac{4}{3}\right)(2^2) = \frac{82.3}{EI}$$

$$\frac{217.3}{EI} + V\,\frac{82.3}{EI} = 0 \rightarrow \boxed{V = -2.64}$$

مثال: مطلوب است تحلیل تیر شکل زیر.



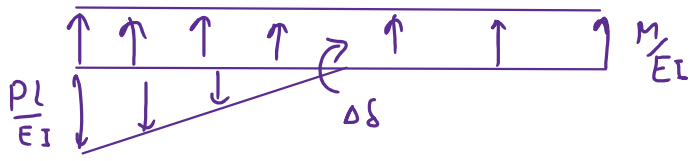


سازگاری تغییر شکل ها

$$\theta_L = \theta_R$$

$$\frac{PL^2}{2EI} - \frac{ML}{EI} = \frac{ML}{EI} \rightarrow \boxed{M = \frac{PL}{4}}$$

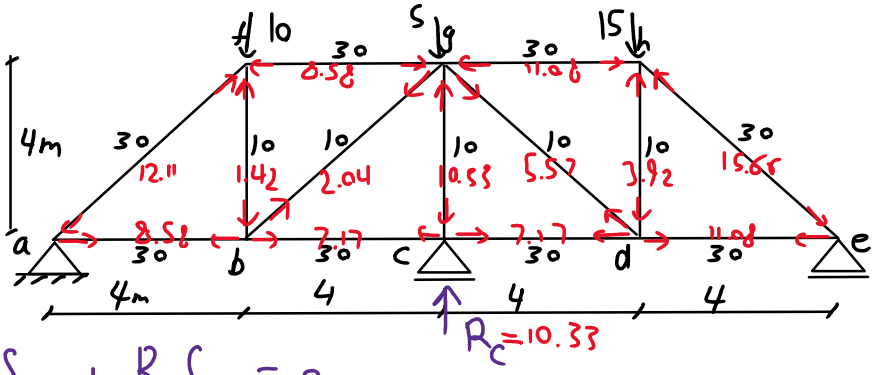
تعادل تیر مزدوج



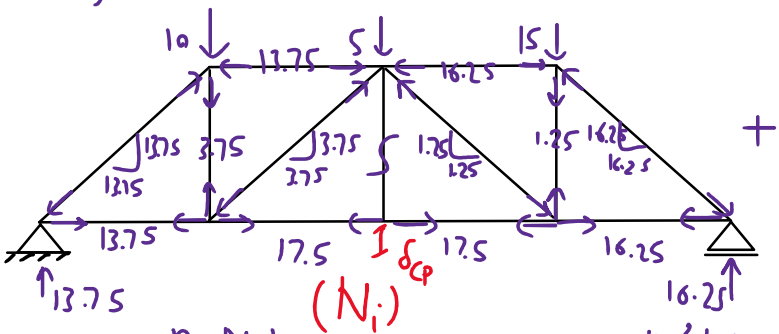
θ	V
δ_1, δ_2	M_1, M_2

$$\Sigma F_y = 0 \rightarrow \frac{M}{EI}(2L) - \frac{1}{2}\left(\frac{PL}{EI}\right)(L) = 0 \rightarrow \boxed{M = \frac{PL}{4}}$$

مثال : خرپای شکل زیر را تحلیل کنید. (مساحت اعضا برحسب cm^2 روی آن نوشته شده است.)

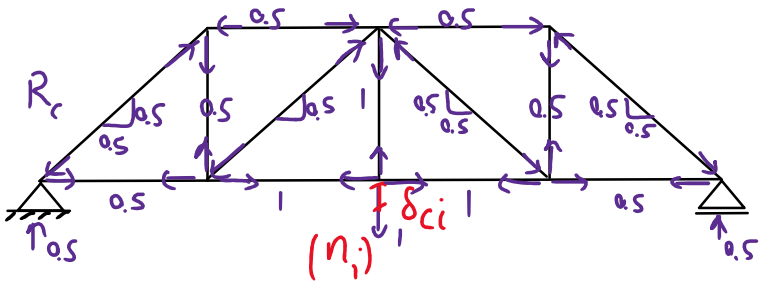


$\delta_{cy} = 0 \rightarrow \delta_{cp} + R_c \delta_{ci} = 0$



(N_i)

+ R_c



(n_i)

$$\delta_{cp} = \sum \frac{n_i N_i L_i}{E_i A_i} \qquad \delta_{ci} = \sum \frac{n_i^2 L_i}{E_i A_i}$$

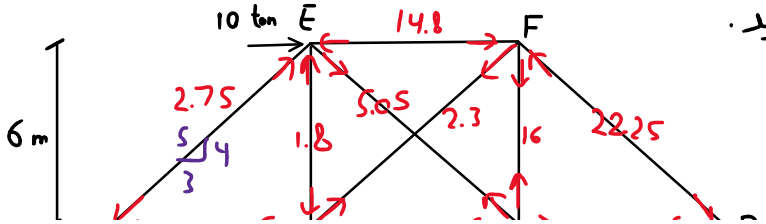
$$\delta_{cp} = \frac{1819.41}{E} \qquad \delta_{ci} = \frac{176.09}{E}$$

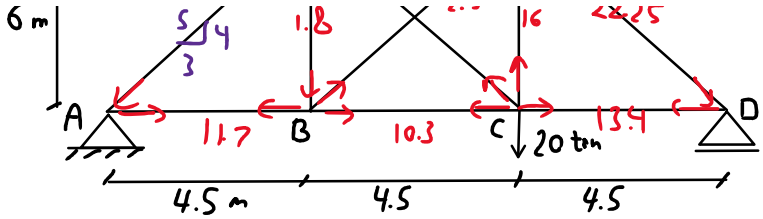
$$\frac{1819.41}{E} + R_c \frac{176.09}{E} = 0$$

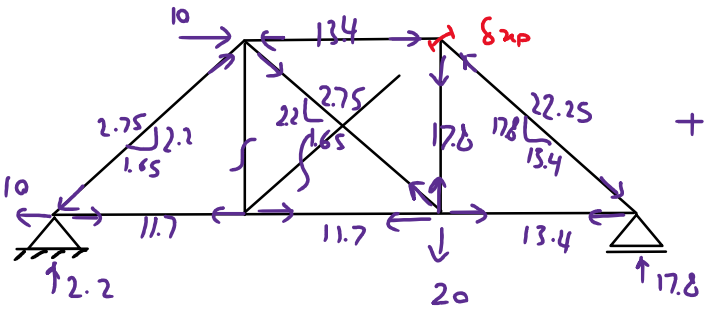
$$\boxed{R_c = -10.33}$$

Element	Ai	Li	ni	Ni	ni*Ni*Li/Ai	ni^2*Li/Ai	Ni + Rc*ni
ab	30	400	0.5	13.75	91.67	3.33	8.58
bc	30	400	1	17.5	233.33	13.33	7.17
cd	30	400	1	17.5	233.33	13.33	7.17
de	30	400	0.5	16.25	108.33	3.33	11.08
fg	30	400	-0.5	-13.75	91.67	3.33	-8.58
gh	30	400	-0.5	-16.25	108.33	3.33	-11.08
af	30	566	-0.71	-19.45	260.54	9.51	-12.11
bf	10	400	0.5	3.75	75.00	10.00	-1.42
bg	10	566	-0.71	-5.3	212.99	28.53	2.04
cg	10	400	1	0	0.00	40.00	-10.33
dg	10	566	-0.71	-1.77	71.13	28.53	5.57
dh	10	400	0.5	1.25	25.00	10.00	-3.92
he	30	566	-0.71	-23	308.09	9.51	-15.66
					1819.41	176.09	**-10.33**

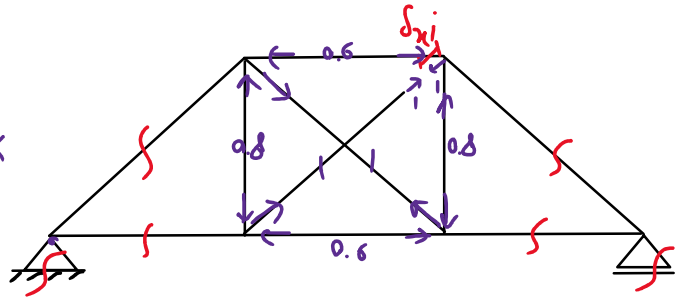
مثال : خرپای شکل زیر را تحلیل کنید.







+ ×



$$\delta_{xp} + X\,\delta_{xi} = 0$$

$$1\times\delta_{xp} = \sum \frac{n_i N_i L_i}{EA} = \frac{1}{EA}\left[0 + (-13.4)(-0.6)(4.5) + (17.8)(-0.8)(6) + (11.7)(-0.6)(4.5) + (2.75)(1)(7.5) + 0\right] = \frac{-59.85}{EA}$$

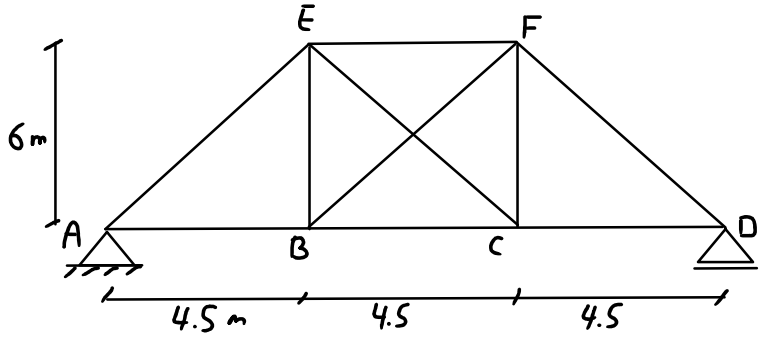
$$1\times\delta_{xi} = \sum \frac{n_i^2 L_i}{EA} = \frac{1}{EA}\left[2\times(-0.6)^2(4.5) + 2\times(-0.8)^2(6) + 2\times(1)^2(7.5)\right] = \frac{25.92}{EA}$$

$$\frac{-59.85}{EA} + X\,\frac{25.92}{EA} = 0 \rightarrow \boxed{X = 2.3 \text{ ton}}$$

مثال: الف) اگر بال خارجی خرپا (AE , EF , FD) به میزان 33°c افزایش دما یابد، نیروی اعضای خرپا را به دست آورید.

ب) درصورتی که علاوه بر تغییر دما، عضو BF در اثر خطای مونتاژ 2 mm کوتاه‌تر اجرا شده باشد، نیروی اعضا چقدر است؟

نشست تکیه‌گاه D به اندازه 3 cm



$\alpha = 1.2\times10^{-5}\ 1/°c$

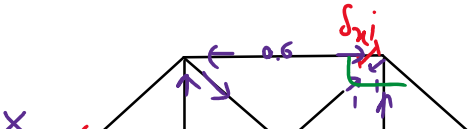
$E = 2\times10^{6}\ kg/cm^2$

$A = 50\ cm^2$

$EA = 100\times10^{6}\ kg = 10^{5}\ ton$

$$\delta_{xp} + X\,\delta_{xi} = 0$$

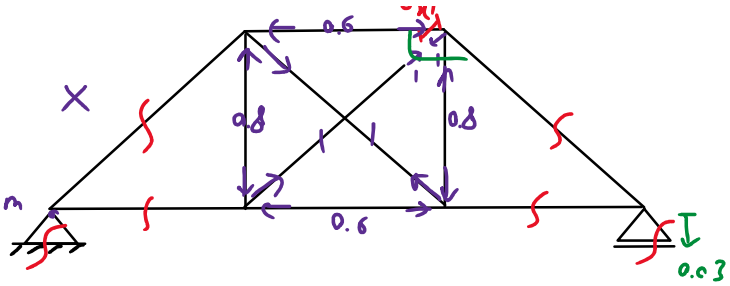
+ a x 0.03



×

$$\delta_{xp} + X\,\delta_{xi} = 0$$

$$1\times\delta_{xp} = \sum n_i(\alpha L_i \Delta T_i) \qquad +0\times0.03$$



$$\delta_{xp} = 0 + 0 + (-0.6)(1.2\times10^{-5}\times4.5\times33) = -0.001\ m = -1\ mm$$

$$1\times\delta_{xi} = \frac{2.25}{EA}\ \frac{ton\cdot m^2}{ton} = \frac{25.92}{10^5} = 0.26\times10^{-3}\ m = 0.26\ mm$$

$$-1 + X(0.26) = 0 \rightarrow \boxed{X = 3.85\ ton}$$

$$1\times\delta_{xp} = (-0.6)(1.2\times10^{-5}\times4.5\times33) + 1\times(-0.002) = -0.001 - 0.002 = -0.003\ m$$

$$-3 + X(0.26) = 0 \rightarrow \boxed{X = 11.5\ ton}$$

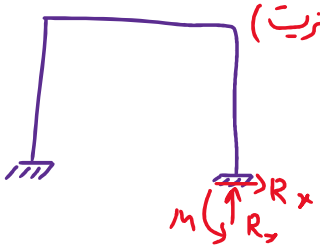
تحلیل سازه‌های نامعین با استفاده از قضیه دوم کاستیلیانو (قضیه انرژی حداقل)

طبق قضیه دوم کاستیلیانو $\frac{\partial U}{\partial P_n} = \delta_n$ است.

در تحلیل سازه‌های نامعین تیرهای اضافه، نیروهای مجهول اضافه و تغییر مکان معلوم (معمولاً برابر صفر) به وجود می‌آورد. بنابراین می‌توان مجهولات اضافی را از معادله $\frac{\partial U}{\partial P_n} = 0$ به دست آورد.

یعنی نیروی مجهول اضافه باید مقداری باشد که انرژی درونی داخلی در سازه حداقل شود.

* در قضیه کاستیلیانو، نشست تکیه‌گاهی، تنش حرارتی و تغییر دما نداریم. (محدودیت روش)
* این روش نسبت به روش سازگاری تغییر شکل‌ها، تعداد انتگرال کمتری دارد. (مزیت)

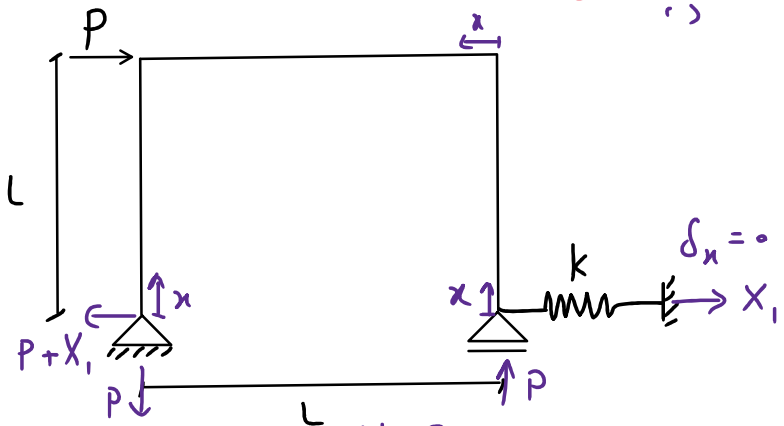


$\delta_x = 0 \rightarrow \frac{\partial U}{\partial R_x} = 0$

$\delta_y = 0 \rightarrow \frac{\partial U}{\partial R_y} = 0$

$\theta = 0 \rightarrow \frac{\partial U}{\partial M} = 0$

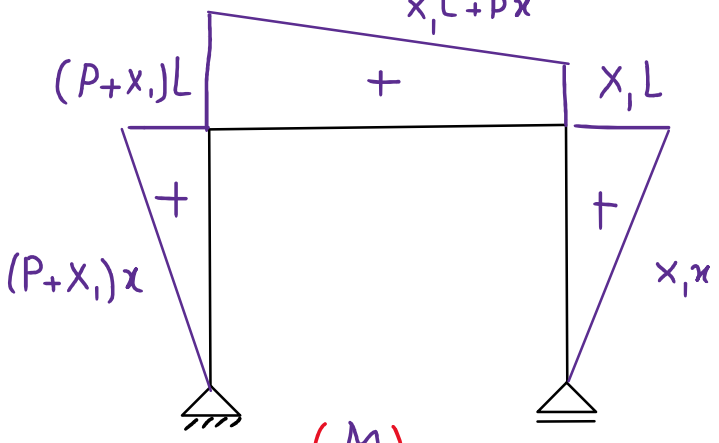
مثال: مطلوب است عکس‌العمل تکیه‌گاهی فنر.

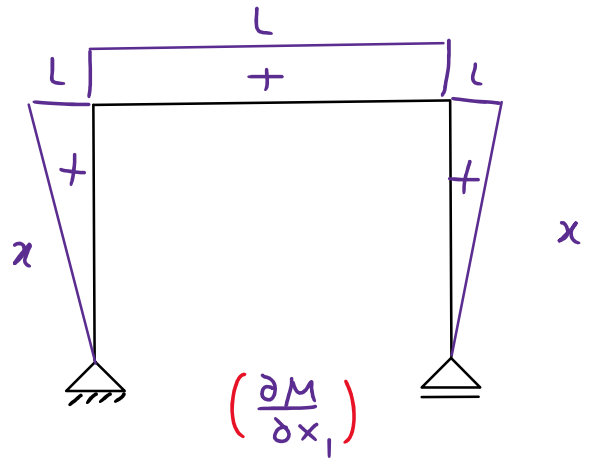


$$\frac{\partial U}{\partial X_1} = 0$$

$$U = \frac{1}{2}\int \frac{M^2}{EI}dx + \frac{1}{2}\frac{F^2}{k}$$

$$\frac{\partial U}{\partial X_1} = \int \frac{M}{EI}\left(\frac{\partial M}{\partial X_1}\right)dx + \frac{F}{k}\frac{\partial F}{\partial X_1} = 0$$





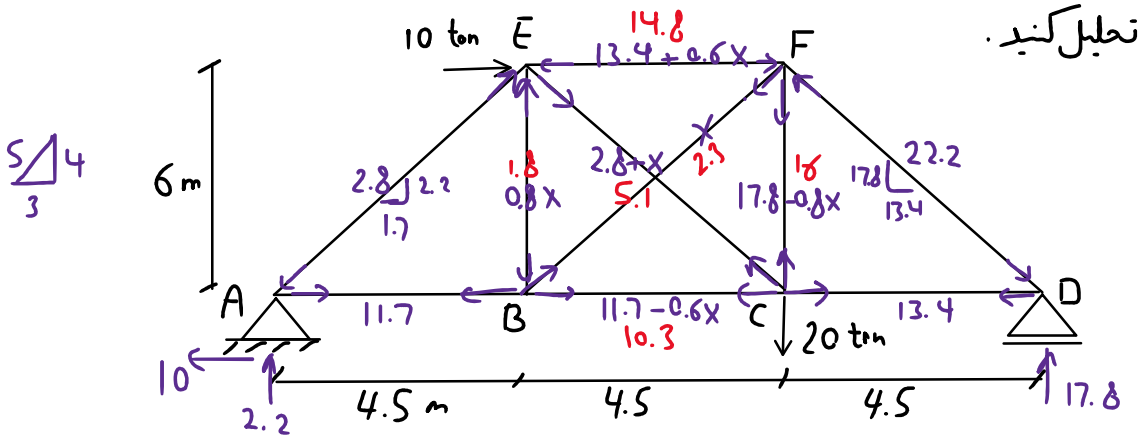
$$\frac{\partial U}{\partial X_1} = \int \frac{M}{EI}\left(\frac{\partial M}{\partial X_1}\right)dx + \frac{F}{k}\frac{\partial F}{\partial X_1} = 0$$

$$\frac{1}{EI}\left[\left(\frac{L}{3}\right)(P+X_1)L\,(L)+\left(\frac{L}{3}\right)(X_1L)(L)+(L)\left(X_1L+\frac{PL}{2}\right)(L)\right]+\left(\frac{X_1}{k}\right)(1)=0$$

$$\frac{1}{EI}\left[\frac{PL^3}{3}+\frac{X_1L^3}{3}+\frac{X_1L^3}{3}+X_1L^3+\frac{PL^3}{2}\right]+\frac{X_1}{k}=0$$

$$\frac{1}{EI}\left[\frac{5}{6}PL^3+\frac{5}{3}X_1L^3\right]+\frac{X_1}{k}=0 \rightarrow \boxed{X_1=\frac{-P}{2+\frac{6}{5}\left(\frac{EI}{L^3k}\right)}}$$

مثال : خرپای شکل زیر را تحلیل کنید.



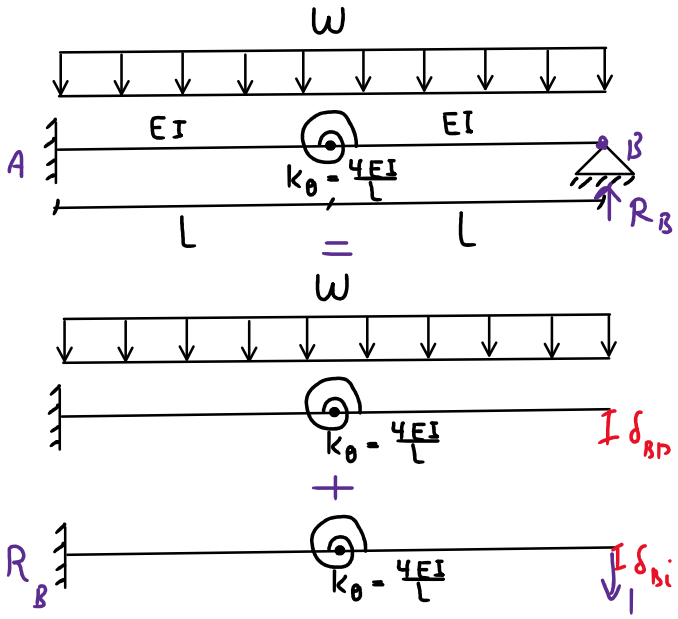
$$\frac{\partial U}{\partial X}=0 \rightarrow U=\frac{1}{2}\sum\frac{N^2L}{EA} \rightarrow \frac{\partial U}{\partial x}=\sum\frac{N}{EA}\left(\frac{\partial N}{\partial X}\right)L=0$$

$$\frac{1}{EA}\Big[(-0.8X)(-0.8)(6)+(11.7-0.6X)(-0.6)(4.5)+(17.8-0.8X)(-0.8)(6)+$$
$$(-(13.4+0.6X))(-0.6)(4.5)+(X)(1)(7.5)+(2.8+X)(1)(7.5)\Big]=0$$

$$-58.86+25.92X=0 \rightarrow \boxed{X=2.3\ \text{ton}}$$

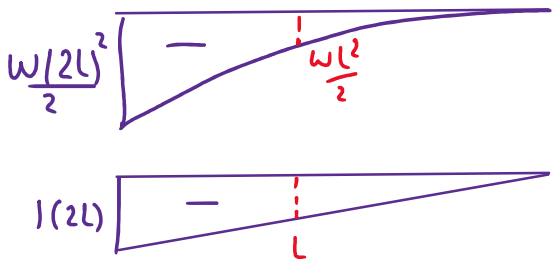
مثال: تیر شکل زیر را به روش‌های مختلف تحلیل کنید.

① سازگاری تغییر شکل‌ها



$\delta_B = 0$

(الف) بار واحد



$\delta_{BP} + R_B \delta_{Bi} = 0$

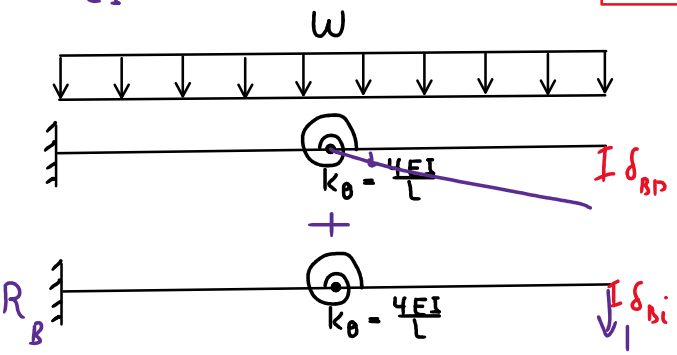
$1 \times \delta_{BP} = \int \frac{m M}{EI} dx + m \frac{M}{k_\theta}$

$1 \times \delta_{BP} = \frac{1}{EI}\left[\frac{2L}{6}\left\{(2WL^2)(-2L) + 4\left(\frac{WL^2}{2}\right)(-L) + 0\right\}\right] + (L)\left(\frac{WL^2}{2k_\theta \to \frac{4EI}{L}}\right) = \frac{17}{8}\frac{WL^4}{EI}$

$1 \times \delta_{Bi} = \frac{1}{EI}\left(\frac{2L}{3}\right)(2L)^2 + \frac{L^2}{K_\theta} = \left(\frac{8}{3} + \frac{1}{4}\right)\frac{L^3}{EI} = \frac{35}{12}\frac{L^3}{EI}$

$\frac{17}{8}\frac{WL^4}{EI} + R_B\left(\frac{35}{12}\frac{L^3}{EI}\right) = 0 \rightarrow$ $\boxed{R_B = \frac{51}{70}WL}$

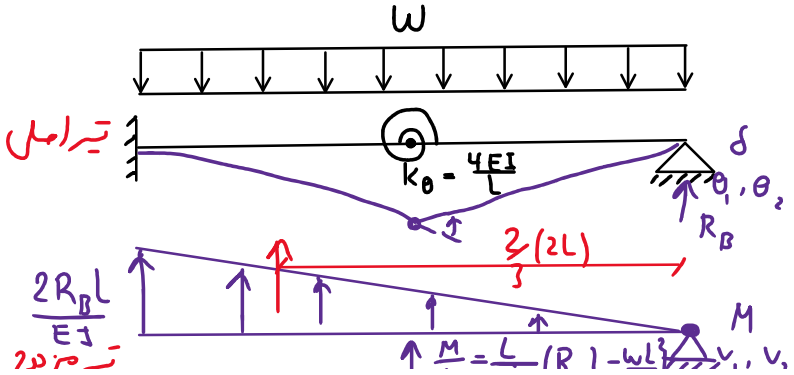
(ب) روابط خطی

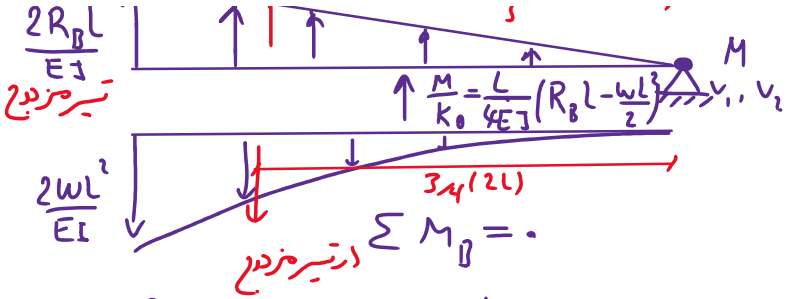


$\delta_{BP} = \frac{W(2L)^4}{8EI} + \frac{WL^2}{2k_\theta}L = \frac{17}{8}\frac{WL^4}{EI}$

$\delta_{Bi} = \frac{1(2L)^3}{3EI} + \frac{L}{k_\theta}(L) = \frac{35}{12}\frac{L^3}{EI}$

② معادلات تعادل تیر مزدوج

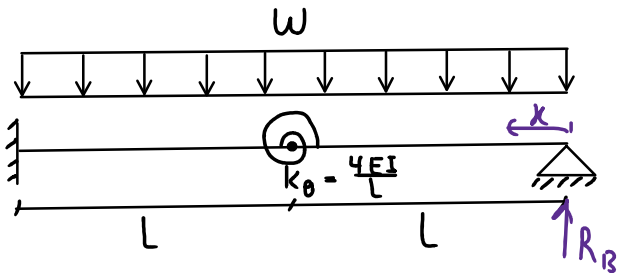




$$\sum M_B = 0$$

$$\frac{1}{2}\left(\frac{2R_B L}{EI}\right)(2L)\left(\frac{2}{3}\times 2L\right) + \frac{L}{4EI}\left(R_B L - \frac{wL^2}{2}\right)(L) - \frac{1}{2}\left(\frac{2wL^2}{EI}\right)(2L)\left(\frac{3}{4}\times 2L\right) = 0$$

$$\frac{8}{3}R_B + \frac{1}{4}R_B - \frac{1}{8}wL - 2wL = 0 \rightarrow \frac{35}{12}R_B = \frac{17}{8}wL \rightarrow \boxed{R_B = \frac{51}{70}wL}$$



② روش کاستیلیانو

$$\delta_B = 0 \rightarrow \frac{\partial U}{\partial R_B} = 0$$

$$U = \frac{1}{2}\int \frac{M^2}{EI}dx + \frac{1}{2}\frac{M^2}{k_\theta}$$

تیر $M = R_B x - \frac{wx^2}{2}$ $\quad \frac{\partial M}{\partial R_B} = x$

فنر $M = R_B L - \frac{wL^2}{2}$ $\quad \frac{\partial M}{\partial R_B} = L$

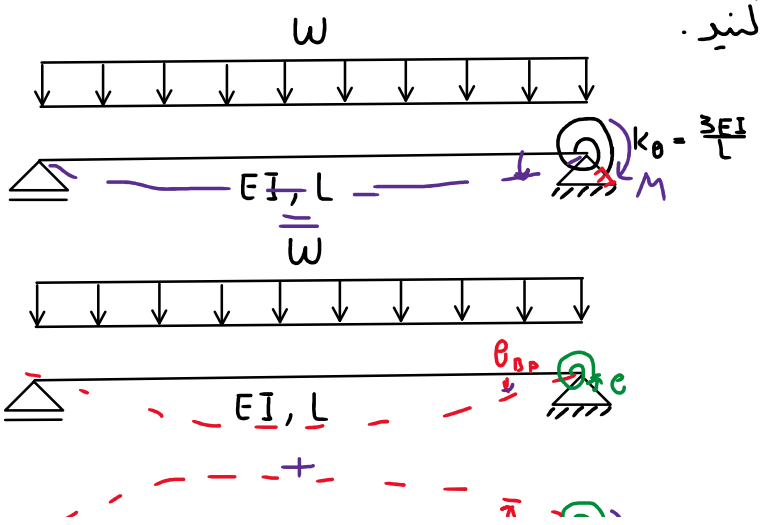
$$\frac{\partial U}{\partial R_B} = \int \frac{M}{EI}\left(\frac{\partial M}{\partial R_B}\right)dx + \frac{M}{k_\theta}\frac{\partial M}{\partial R_B} = 0$$

$$\frac{1}{EI}\int_0^{2L}\left(R_B x - \frac{wx^2}{2}\right)(x)\,dx + \frac{L}{4EI}\left(R_B L - \frac{wL^2}{2}\right)(L) = 0$$

$$\frac{1}{EI}\left(R_B\frac{x^3}{3} - \frac{wx^4}{8}\right)\Big]_0^{2L} + \frac{1}{4}R_B\frac{L^3}{EI} - \frac{1}{8}\frac{wL^4}{EI} = 0 \rightarrow \frac{8}{3}R_B L^3 - 2wL^4 + \frac{1}{4}R_B L^3 - \frac{1}{8}wL^4 = 0$$

$$\boxed{R_B = \frac{51}{70}wL}$$

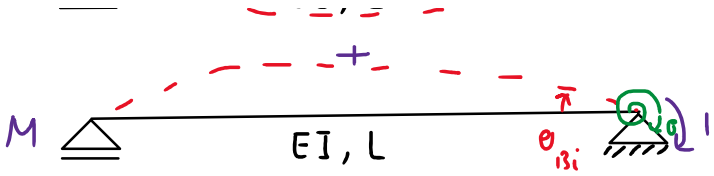
مثال: تیر شکل زیر را به روش های مختلف تحلیل کنید.



① سازگاری تغییر شکل ها

$\theta = 0$

$\theta_{beam} = \theta_{spring}$



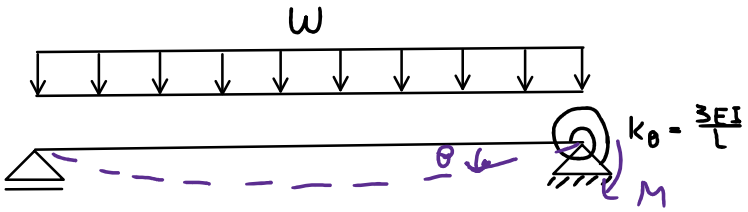
$$1\times\theta_{BP}=\int\frac{mM}{EI}dx=\left(\frac{1}{3}\right)\left(\frac{wL^2}{8EI}\right)(-1)=\frac{-wL^3}{24EI}$$

$$1\times\theta_{Bi}=\int\frac{m^2}{EI}dx=\left(\frac{L}{3}\right)(-1)^2=\frac{L}{3EI}$$

$$-\frac{wL^3}{24EI}+\frac{ML}{3EI}=\frac{-ML}{3EI}\rightarrow\boxed{M=\frac{wL^2}{16}}$$

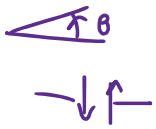
$$-\frac{wL^2}{24EI}+M\left(\frac{L}{3EI}+\frac{1}{\frac{3EI}{L}}\right)=0$$

$$M=\frac{wL^2}{16}$$

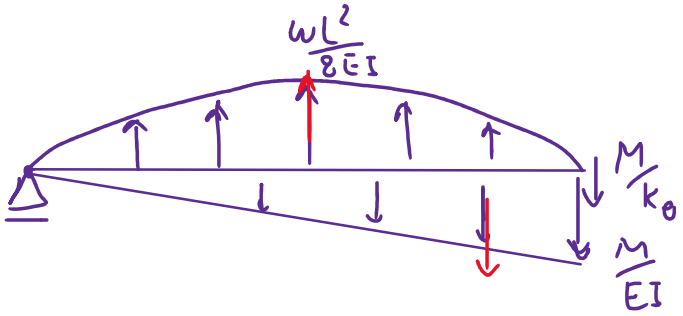


$\delta=0$

$\theta=\frac{M}{k_\theta}$



② تیر مزدوج



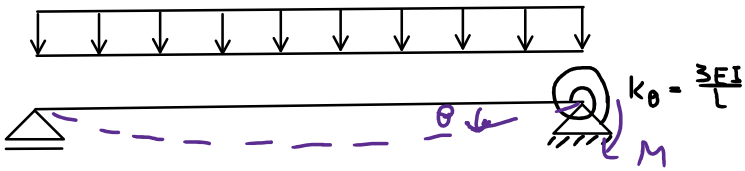
$M=0$

$V=\frac{M}{k_\theta}$

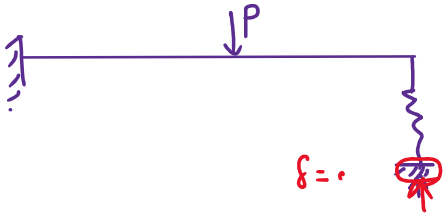
$$\frac{2}{3}\left(\frac{wL^2}{8EI}\right)(L)\left(\frac{L}{2}\right)-\frac{M}{k_\theta}(L)-\frac{1}{2}\left(\frac{M}{EI}\right)(L)\left(\frac{2}{3}L\right)=0$$

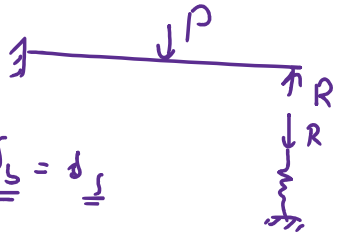
$$\frac{wL^4}{24}-\frac{ML^2}{3}-\frac{ML^2}{3}=0\rightarrow\boxed{M=\frac{wL^2}{16}}$$

② روش توزیع لنگر (کراس)



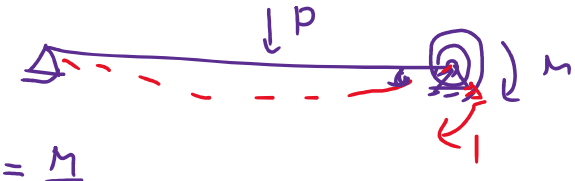
$$M=\frac{wL^2}{8}\times\frac{1}{2}=\frac{wL^2}{16}$$



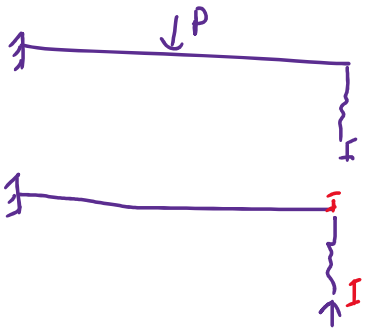


$$-\frac{Pl^3}{3}+\frac{Rl^3}{3}=R$$

$$-\frac{Pl^3}{3EI} \pm \frac{Rl^3}{3EI} = -\frac{R}{k} \qquad R\left(\frac{1}{k_s} + \frac{1}{k_1}\right)$$



$$\frac{Pl^2}{EI} - \frac{Ml}{EI} = \frac{M}{k_\theta}$$



Influence Line1

Thursday, January 18, 2024 10:10

فصل ششم :

خط تأثیر سازه‌های معین

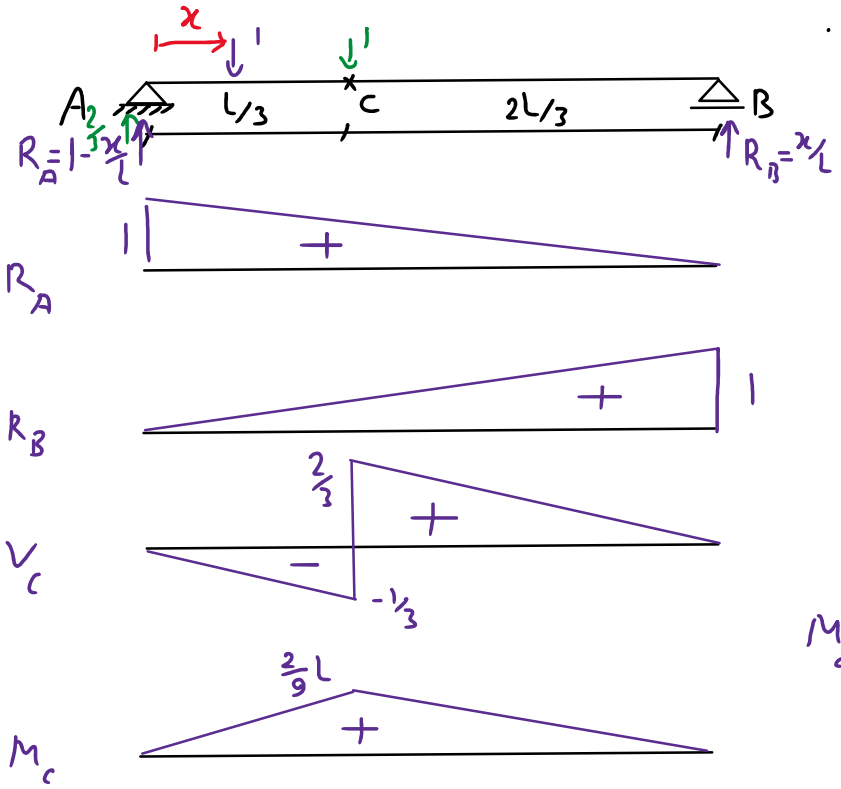
خط تأثیر تابعی است که مقدار یک نیرو (عکس‌العمل تکیه‌گاهی R، نیروی برشی V، لنگر خمشی M) را وقتی بار واحد در نقاط مختلف قرار گرفته است، نشان می‌دهد.

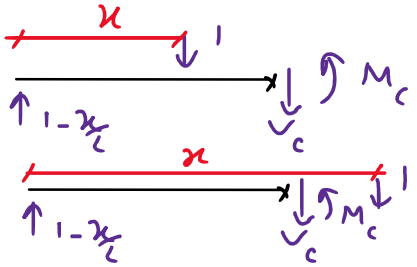
برای رسم خط تأثیر سازه‌های معین دو روش وجود دارد:

۱- نوشتن معادلات تعادل

۲- روش کار مجازی (اصل مولر-برسلاو)

مثال: خط تأثیر R_A، R_B، V_c و M_c را رسم نمایید.





$$V_c = \begin{cases} 1 - \frac{x}{L} & 0 \le x \le \frac{L}{3} \\ -\frac{x}{L} & \frac{L}{3} < x \le L \end{cases}$$

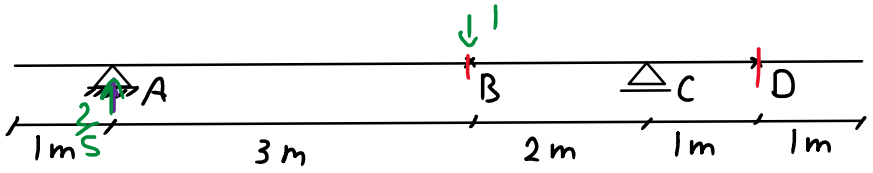
$$M_c = \begin{cases} (1-\frac{x}{L})(\frac{L}{3}) = \frac{2}{3}x & 0 < x < \frac{L}{3} \\ (1-\frac{x}{L})(\frac{L}{3}) = \frac{1}{3}(L-x) & \frac{L}{3} < x \le L \end{cases}$$

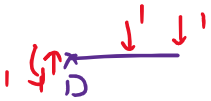
* مقدار خط تأثیر R_D، V_D و M_D در هر نقطه دلخواه x، نشان دهنده عکس‌العمل تکیه‌گاهی، نیروی برشی و لنگر خمشی نقطه ثابت D روی تیر است، هنگامی که بار واحد در نقطه x قرار داشته باشد.

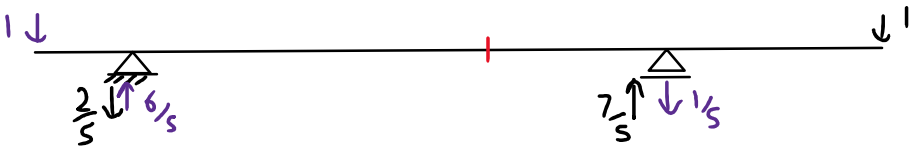
* خط تأثیر سازه‌های معین به صورت چند خطی است.

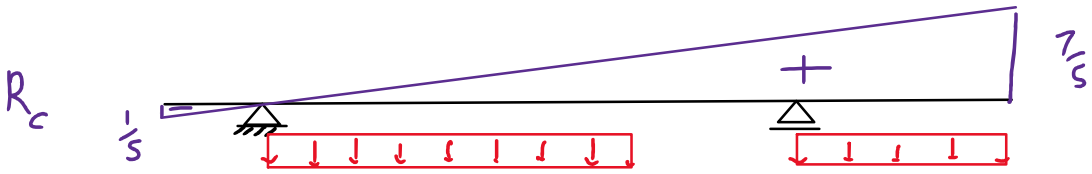
* برای رسم خط تأثیر به روش معادلات تعادل، لازم نیست به صورت بالا تابع نوشته شود، بلکه تعیین نقاط کلیدی و وصل کردن آنها کافی است.

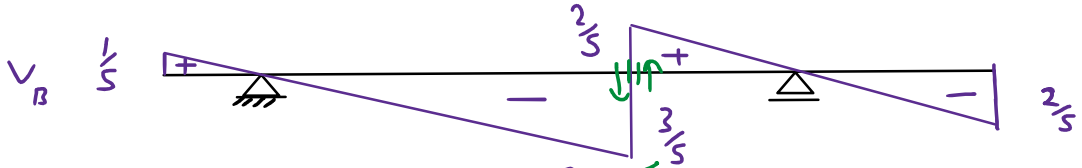
مثال: مطلوب است رسم خط تاثیر R_A ، R_C ، V_B ، M_B ، V_D ، M_D .

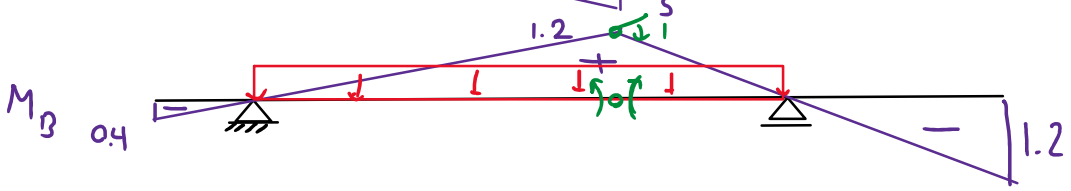


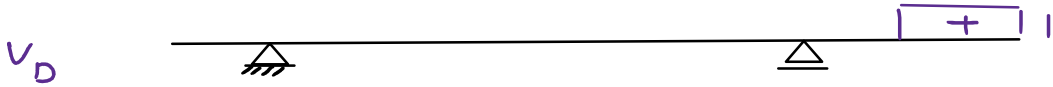


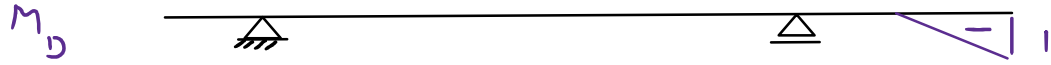












Influence Line2

Thursday, December 28, 2023

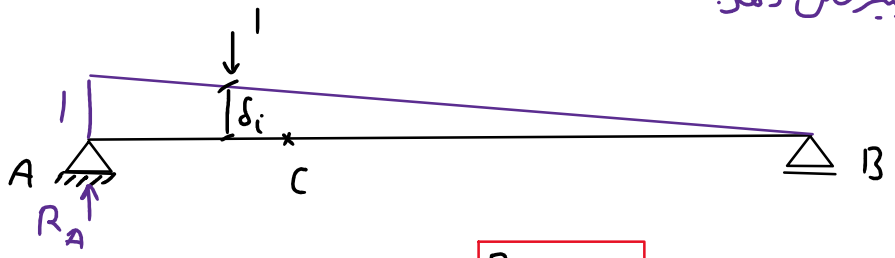
رسم خط تأثیر به روش کار مجازی

در سال ۱۸۸۶ میلادی، مولر- برسلاو یک تکنیک قدی برای رسم خط تأثیر ارائه داد.

اصل مولر- برسلاو:

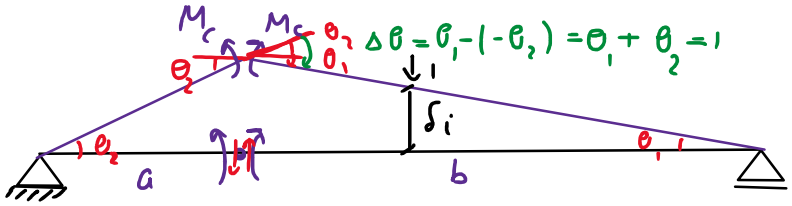
خط تأثیر یک نیرو برابر است با تابع تغییر شکل ساده وقتی که نیروی مورد نظر از سازه حذف شده و سازه آزاد شده تحت اثر همان نیرو تغییر شکل دهد.

(R_A)



$$R_A(1) - 1 \times \delta_i = 0 \rightarrow \boxed{R_A = \delta_i}$$

(M_c)

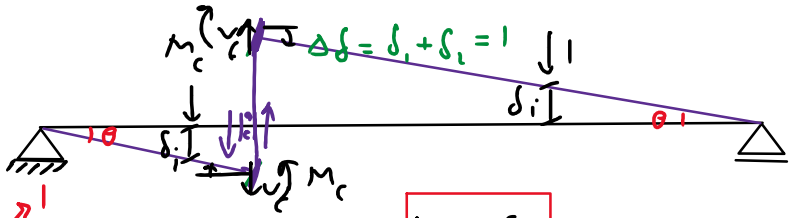


$$\frac{y}{a} + \frac{y}{b} = 1$$

$$y = \frac{ab}{a+b}$$

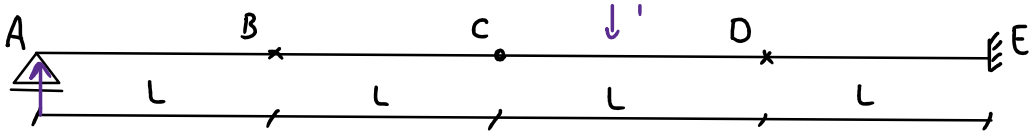
$$M_c(\Delta\theta) - 1 \times \delta_i = 0 \rightarrow \boxed{M_c = \delta_i}$$

(V_c)

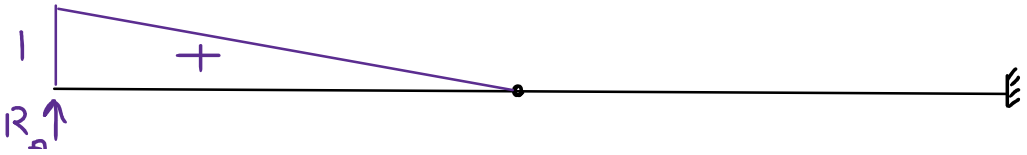


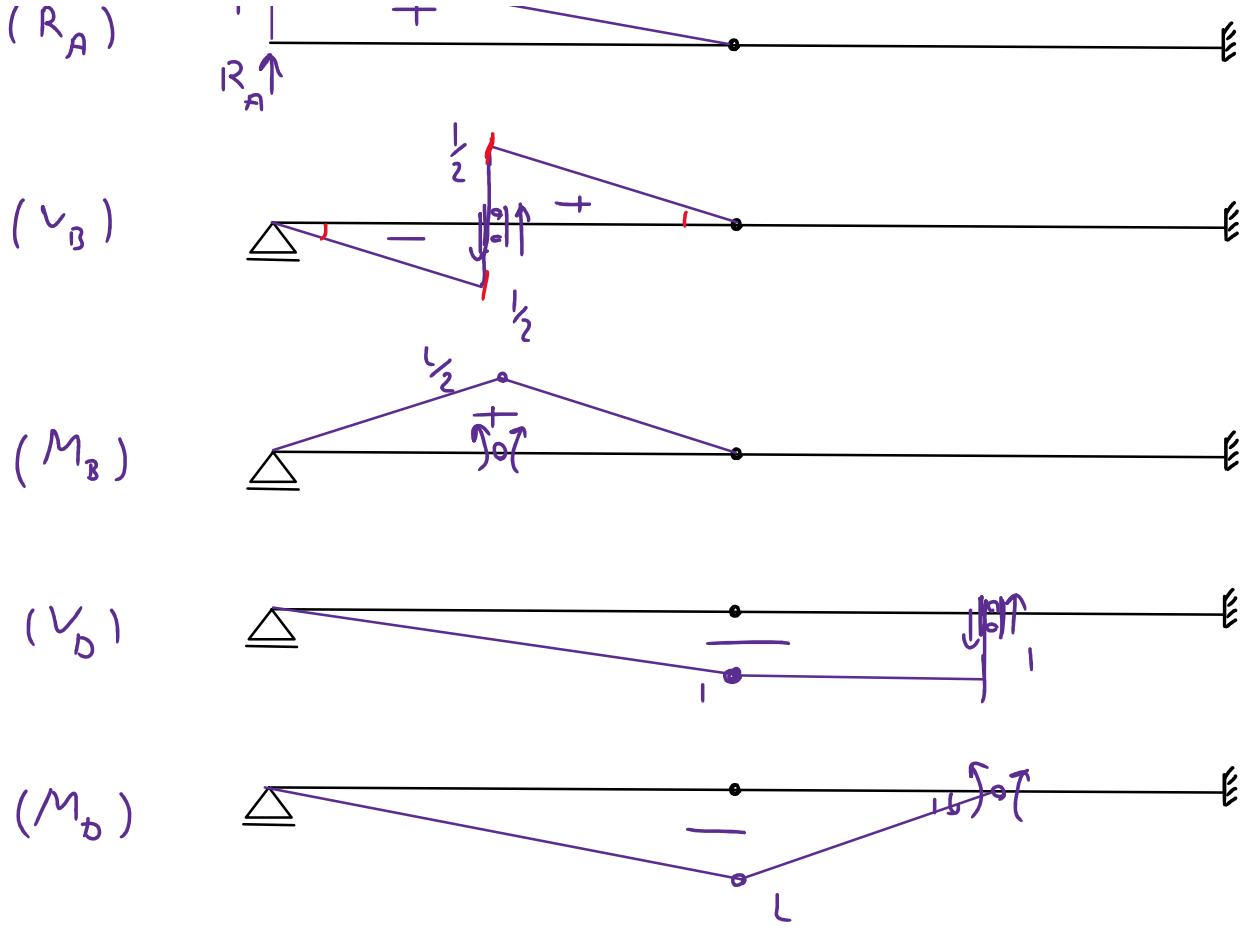
$$M_c\theta - 1 \times \theta + V_c(\Delta\delta) - 1 \times \delta_i = 0 \rightarrow \boxed{V_c = \delta_i}$$

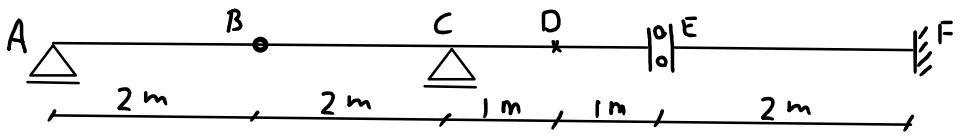
مثال: خط تاثیر R_A، V_B، M_B، V_D و M_D را رسم کنید.



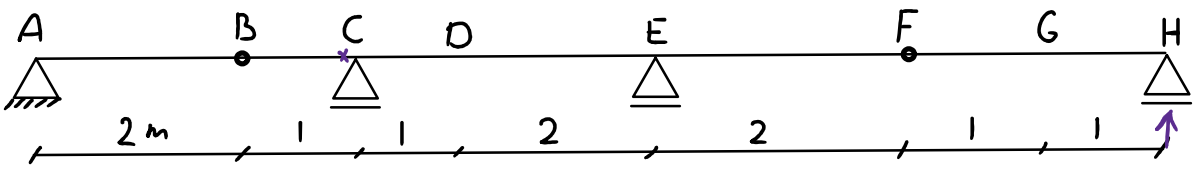
(R_A)

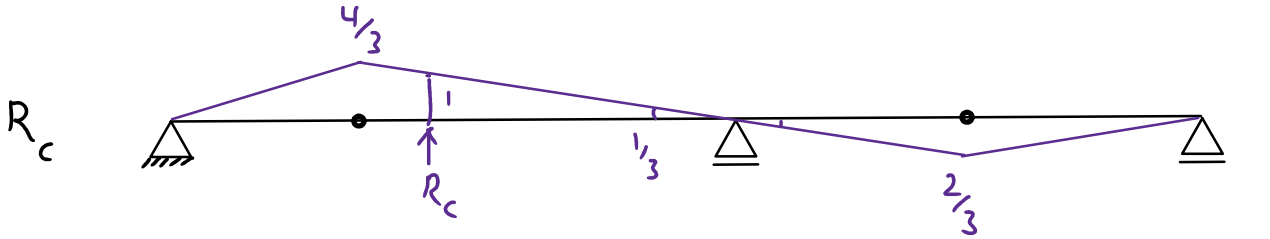


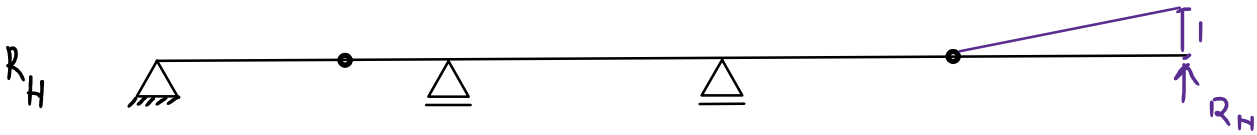

(R_A)
R_A
(V_B)
1/2
1/2
(M_B)
L/2
(V_D)
1
1
(M_D)
L

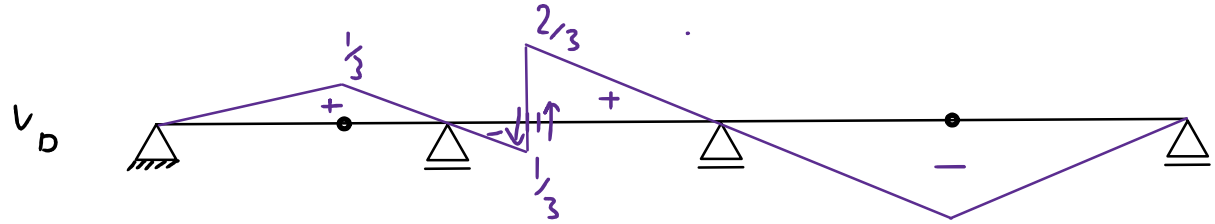

A
B
C
D
E
F
2 m
2 m
1 m
1 m
2 m

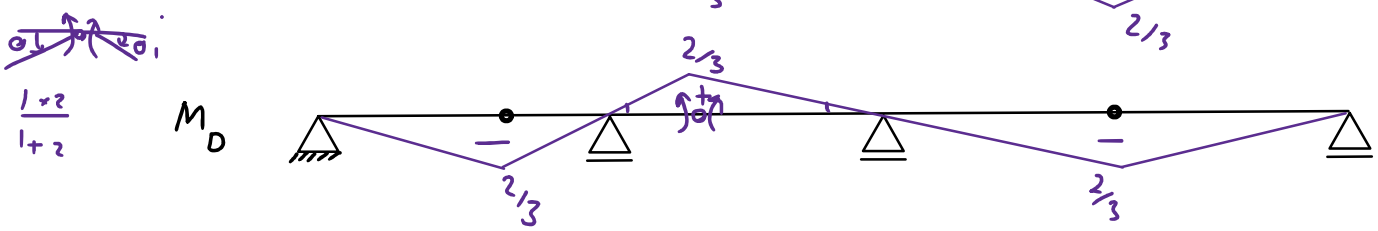
مثال: در تیر شکل زیر، خطوط تأثیر R_c ، R_H ، V_D ، M_D ، M_c ، V_{CR} ، V_{CL} و V_F ، M_G را رسم نمایید.

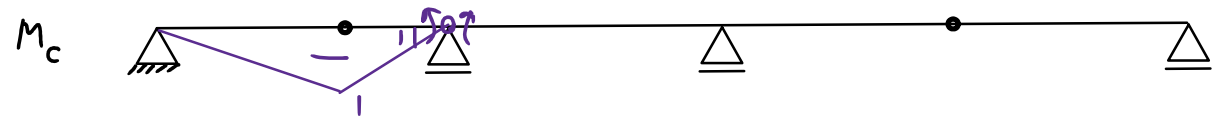


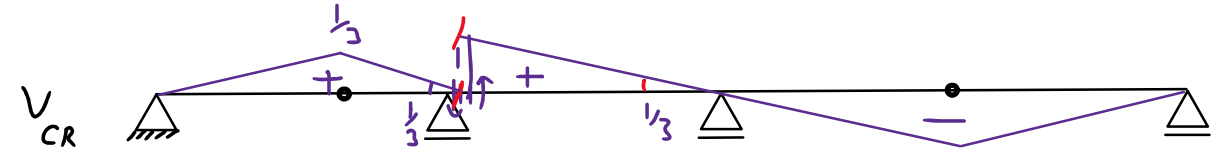


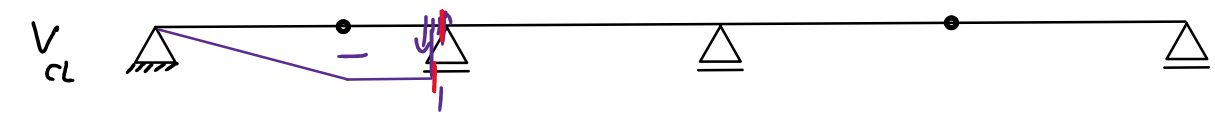


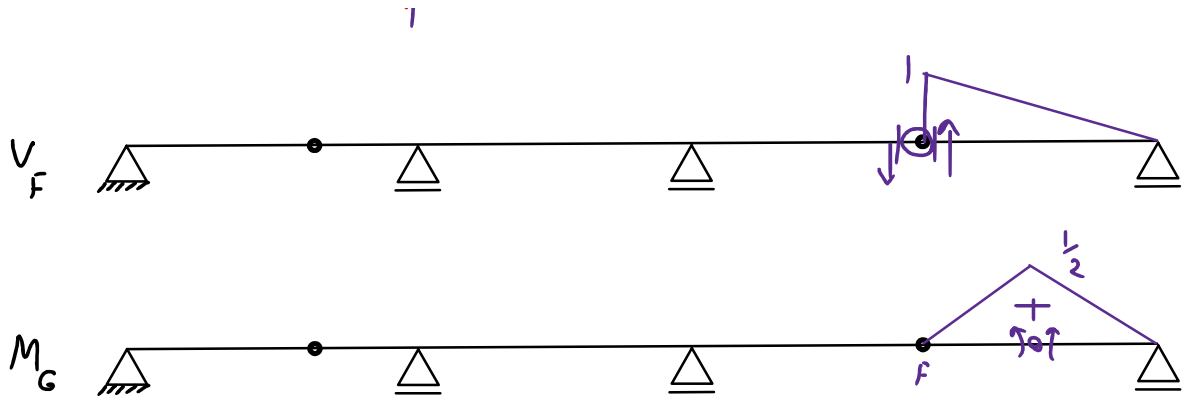




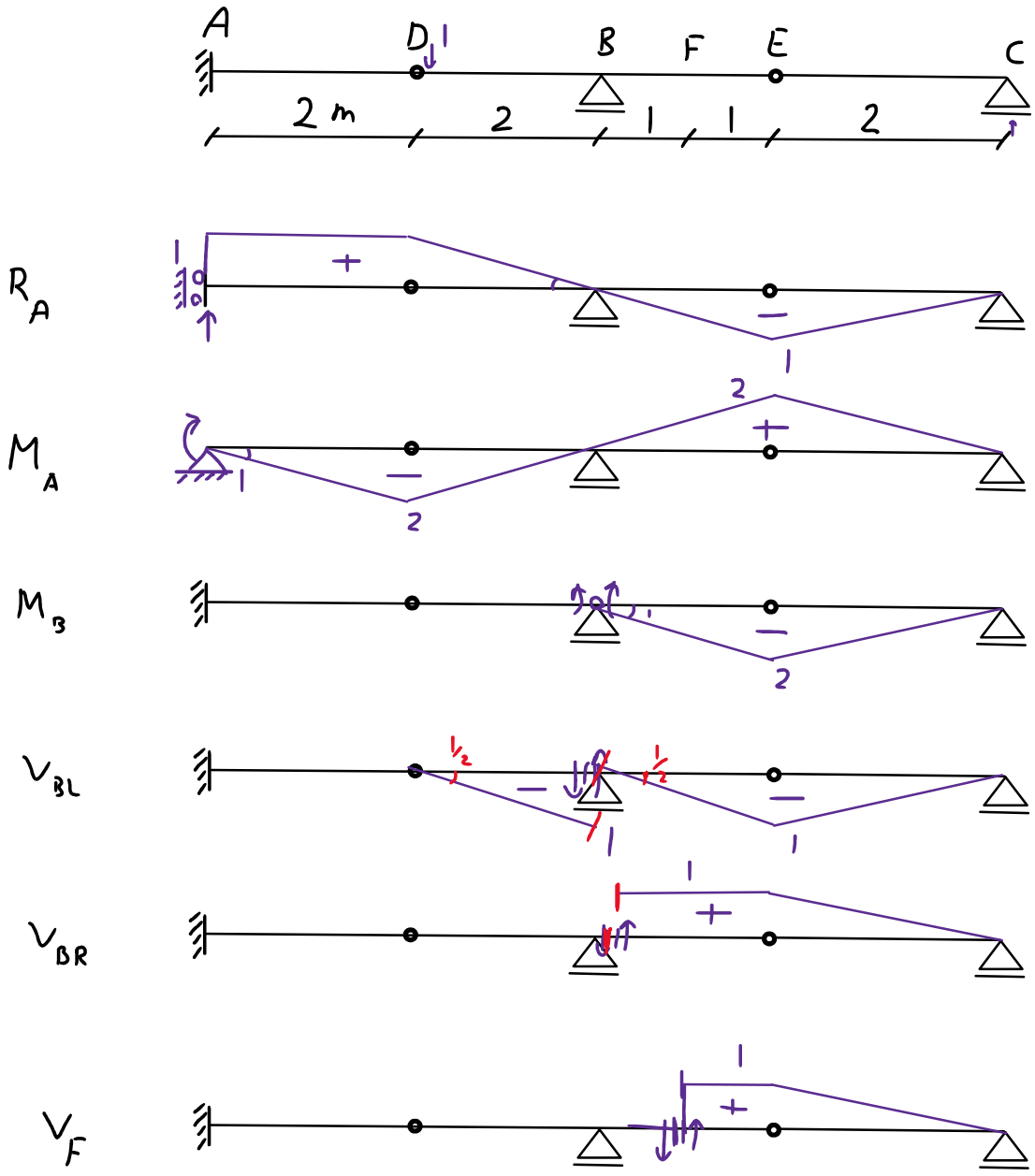




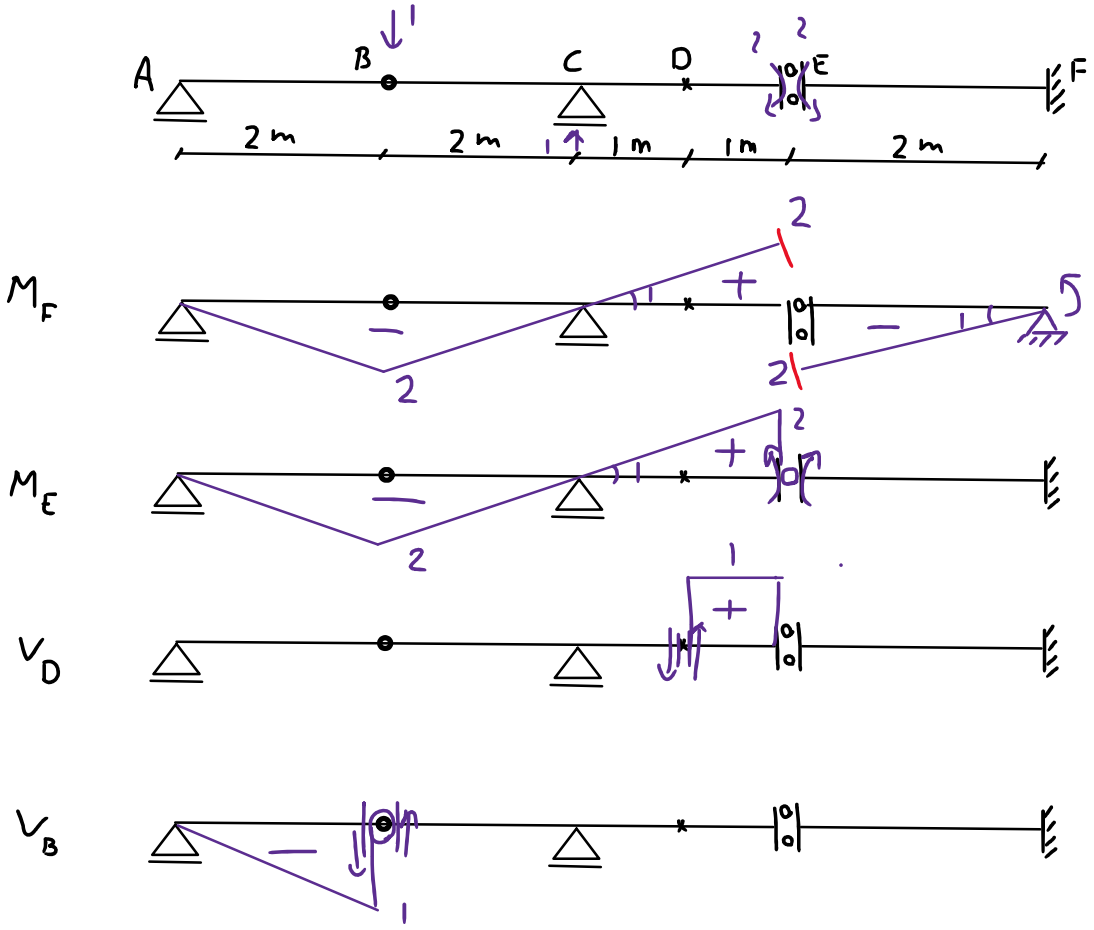




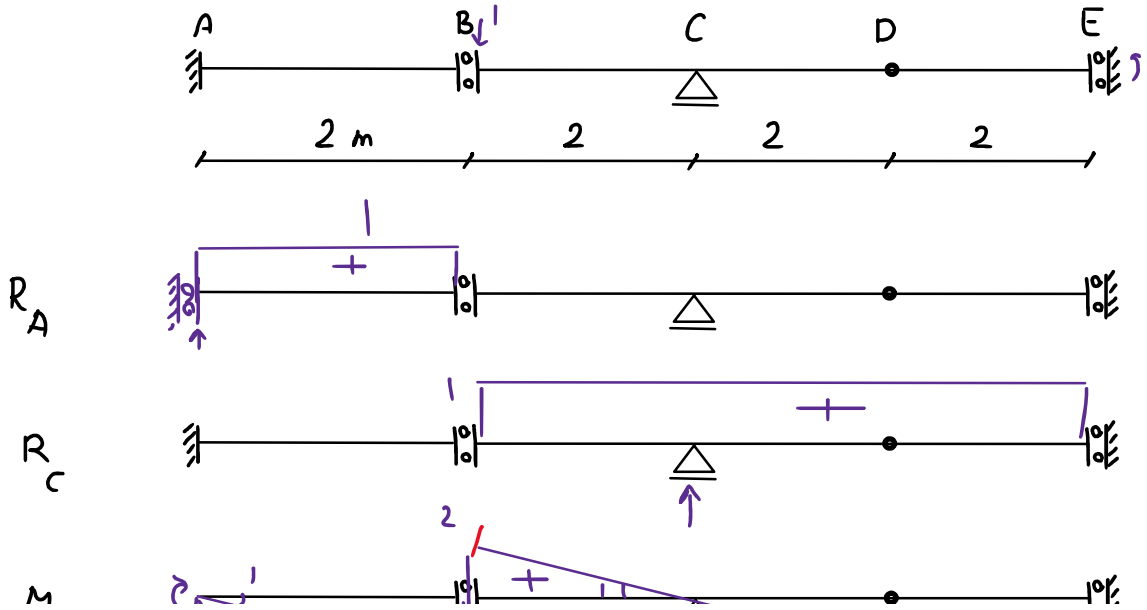
مثال: خطوط تأثیر R_A، M_A، M_B، V_{BL}، V_{BR} و V_F را رسم کنید.

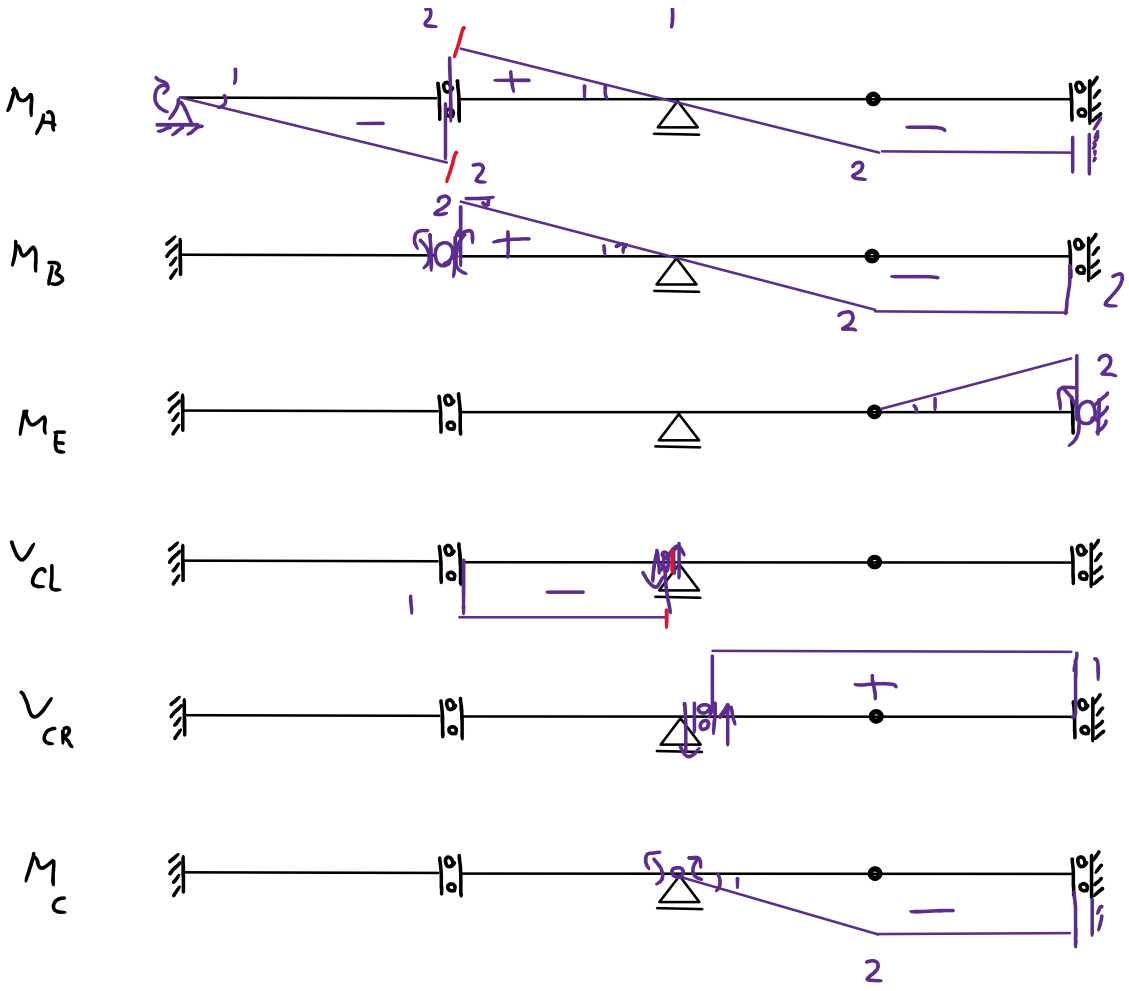


مثال: خطوط تأثیر M_F ، M_E ، V_D و V_B را رسم کنید.



مثال: خطوط تأثیر R_A ، R_C ، M_A ، M_B ، M_E ، V_{CL} ، V_{CR} و M_C را رسم کنید.

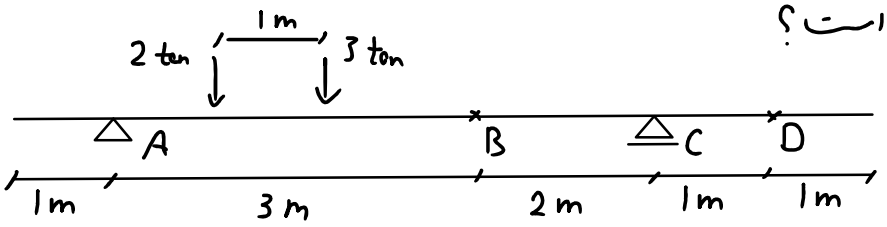



M_A
M_B
M_E
V_{CL}
V_{CR}
M_C

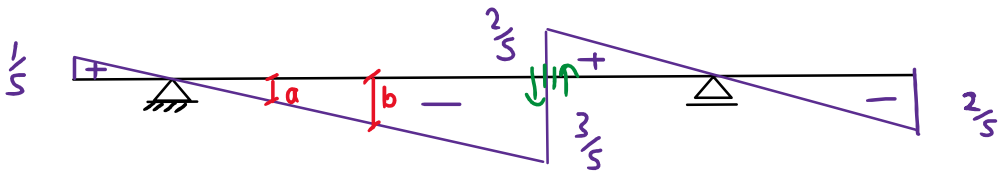
Influence Line4

Sunday, February 11, 2024 13:45

مثال: اگر دو بار مطابق شکل زیر روی تیر حرکت کند، ماکزیمم لنگر و برش در نقطه B چقدر است؟



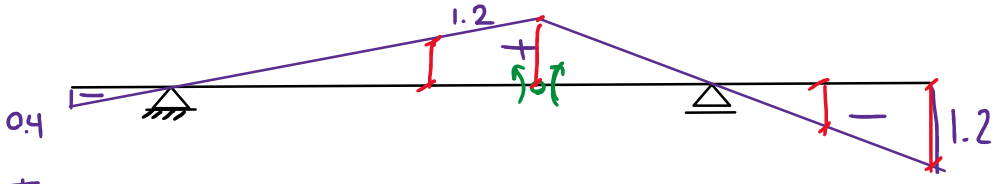
V_B



$V_B = 2a + 3b$

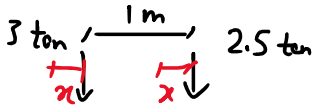
B در 3 ton $V_{max} = 3(-0.6) + 2(-0.4) = -2.6$ ton

M_B

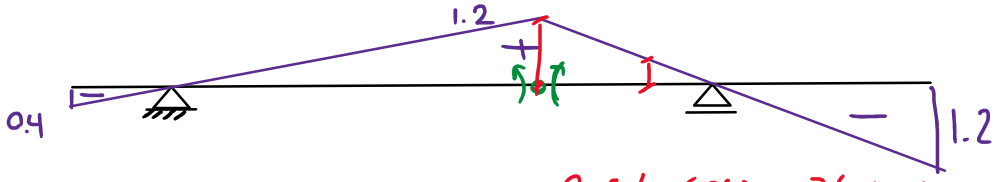


B در 3 ton $M^+_{max} = 2(0.8) + 3(1.2) = 5.2$ t.m

در انتهای تیر 3 ton $M^-_{max} = 2(-0.6) + 3(-1.2) = -4.8$ t.m



M_B

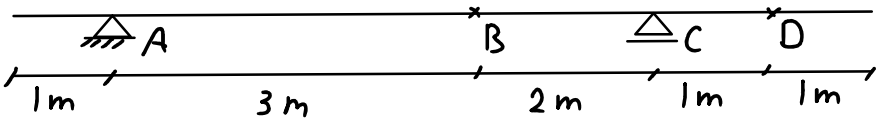


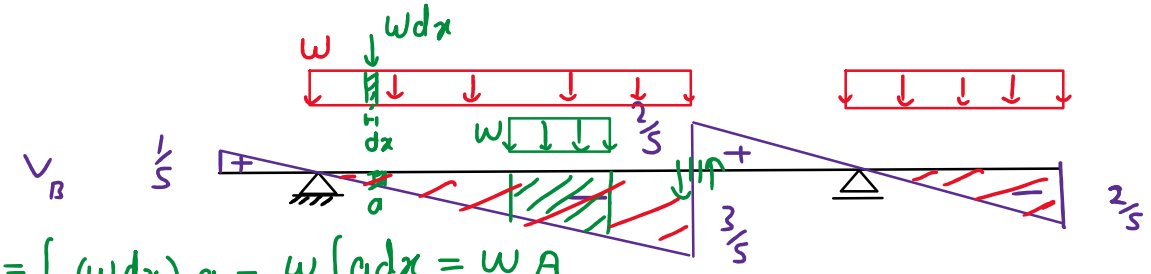
$M = 3(1.2) + 2.5(0.6) = 5.1$ t.m

$2.5(0.6x) - 3(0.4x) = 0.3x$

B (زیر 2.5 ton) $M^+_{max} = 3(0.8) + 2.5(1.2) = 5.4$ t.m

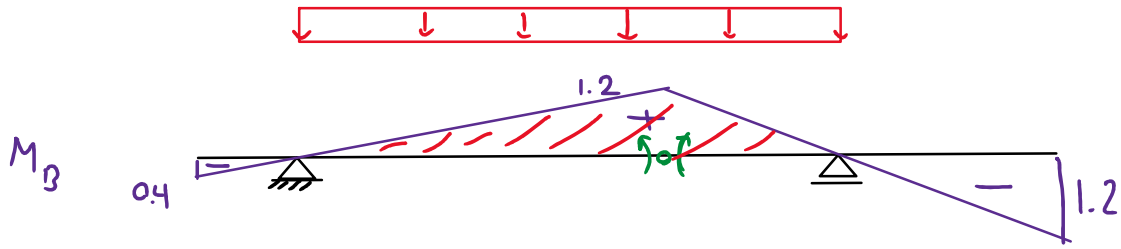
مثال: اگر بار گسترده 2 ton/m با طول متغیر روی تیر وارد شود، حداکثر لنگر و برش نقطه B را به دست آورید.



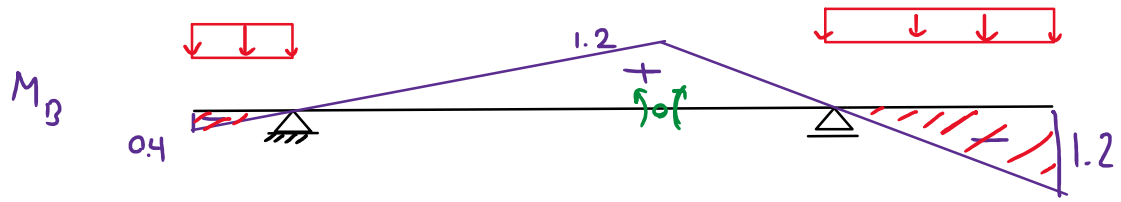


$$V = \int (w\,dx)\,a = w\int a\,dx = w\,A$$

$$V_{max} = w\left[\frac{1}{2}\left(\frac{3}{5}\right)(3) + \frac{1}{2}\left(\frac{2}{5}\right)(2)\right] = 2.6 \text{ ton}$$

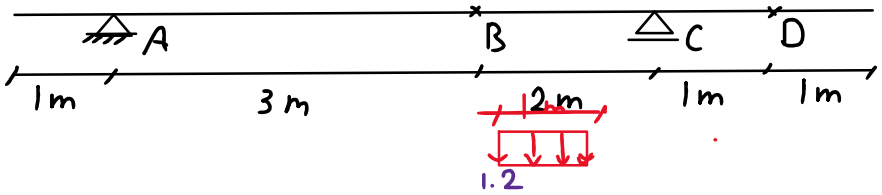


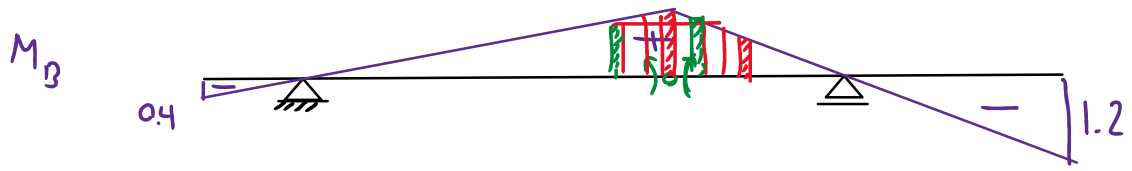
$$M^{+}_{max} = 2\left[\frac{1}{2}(1.2)(5)\right] = 6 \text{ t.m}$$

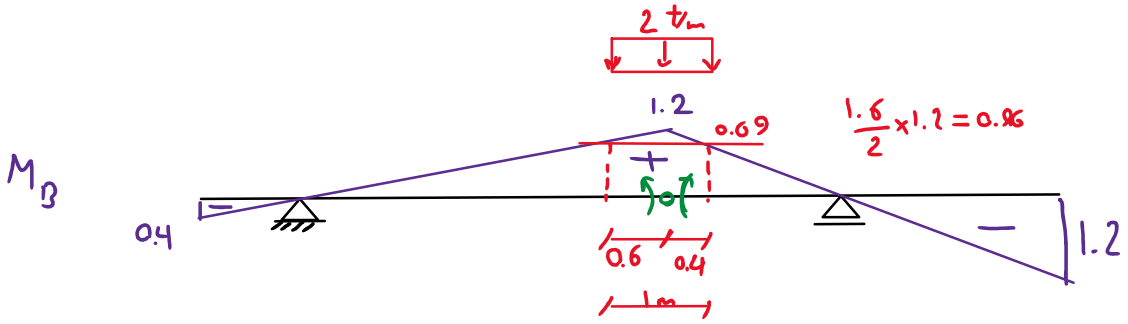


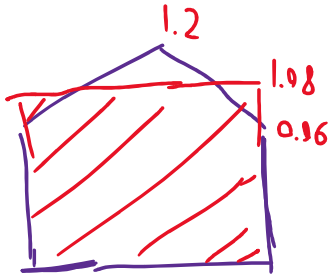
$$M^{-}_{max} = 2\left[\frac{1}{2}(0.4)(1) + \frac{1}{2}(1.2)(2)\right] = 2.8 \text{ t.m}$$

مثال: اگر بار گسترده 2 ton/m به طول 1 m روی تیر وارد شود، حداکثر لنگر و برش نقطه B را به دست آورید.



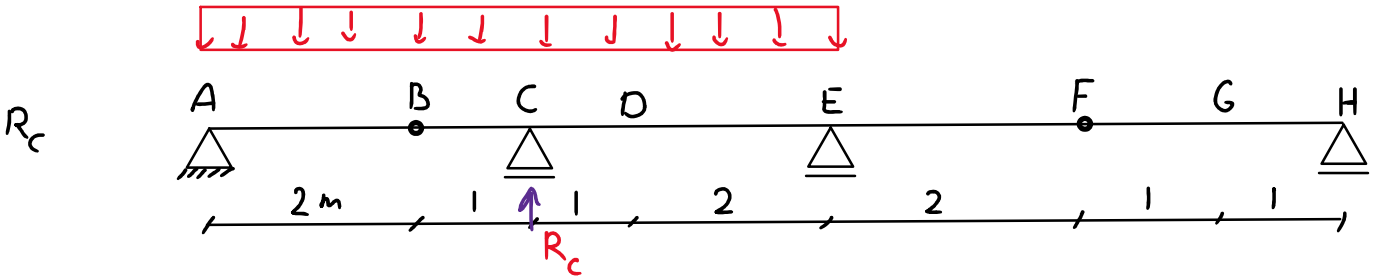


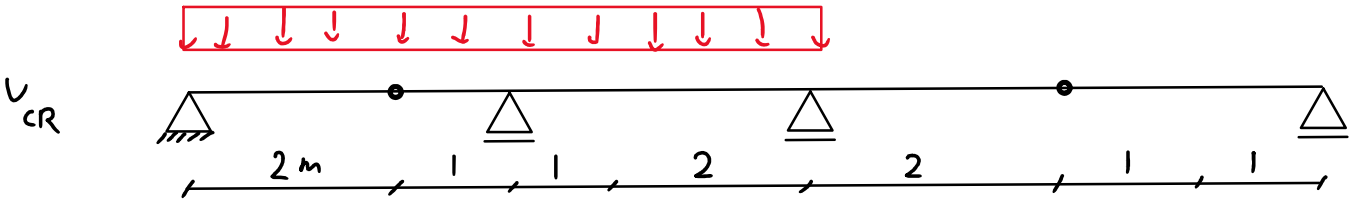


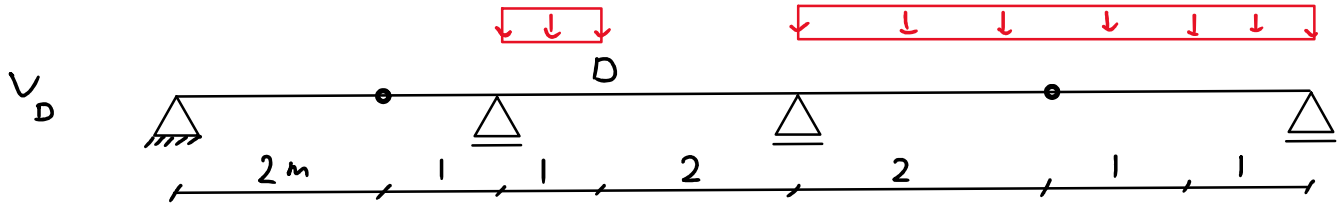


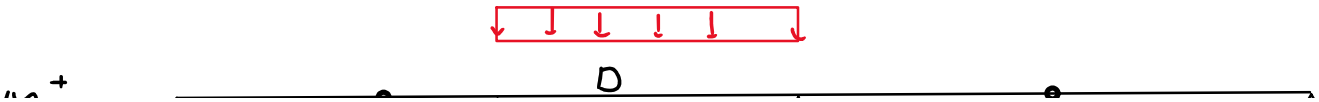
$M_{max} = \omega A = 2 \left[1.08 \times 1 \right] = 2.16$ t.m

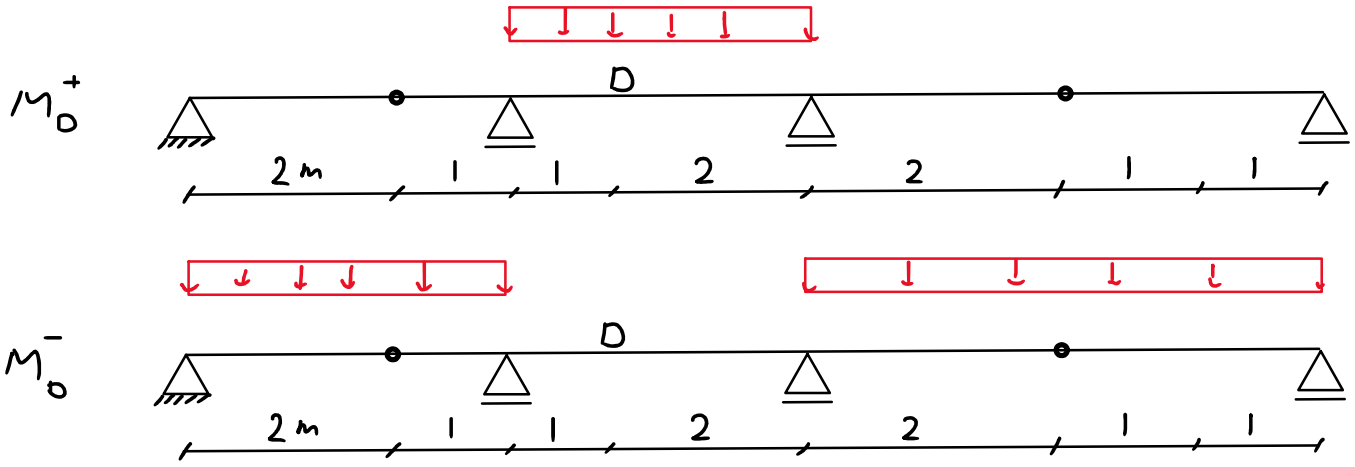
مثال: ماکزیمم R_C ، V_{CR} ، M_D و V_D در کدام حالت توزیع بار گسترده زنده اتفاق می‌افتد؟



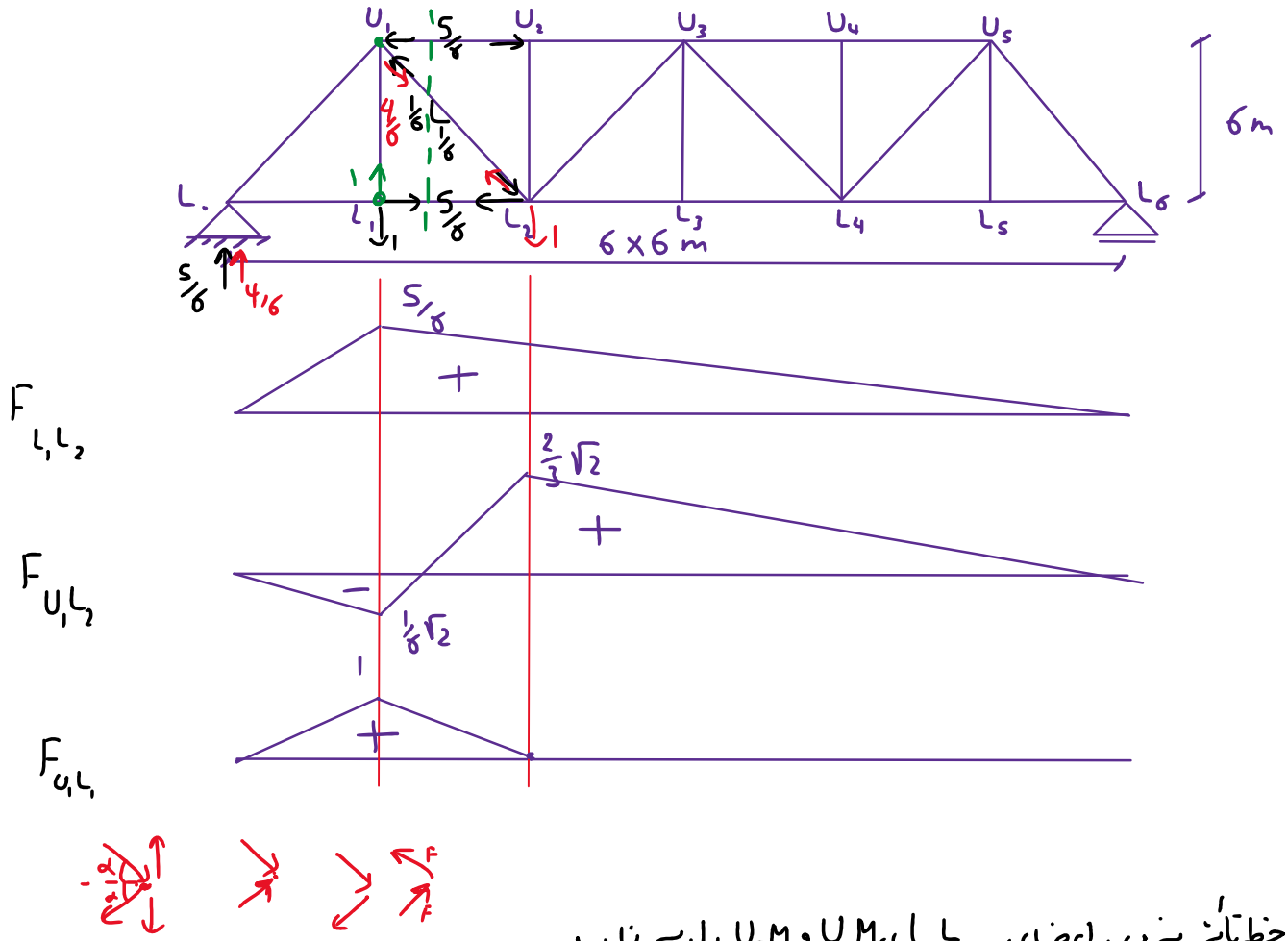




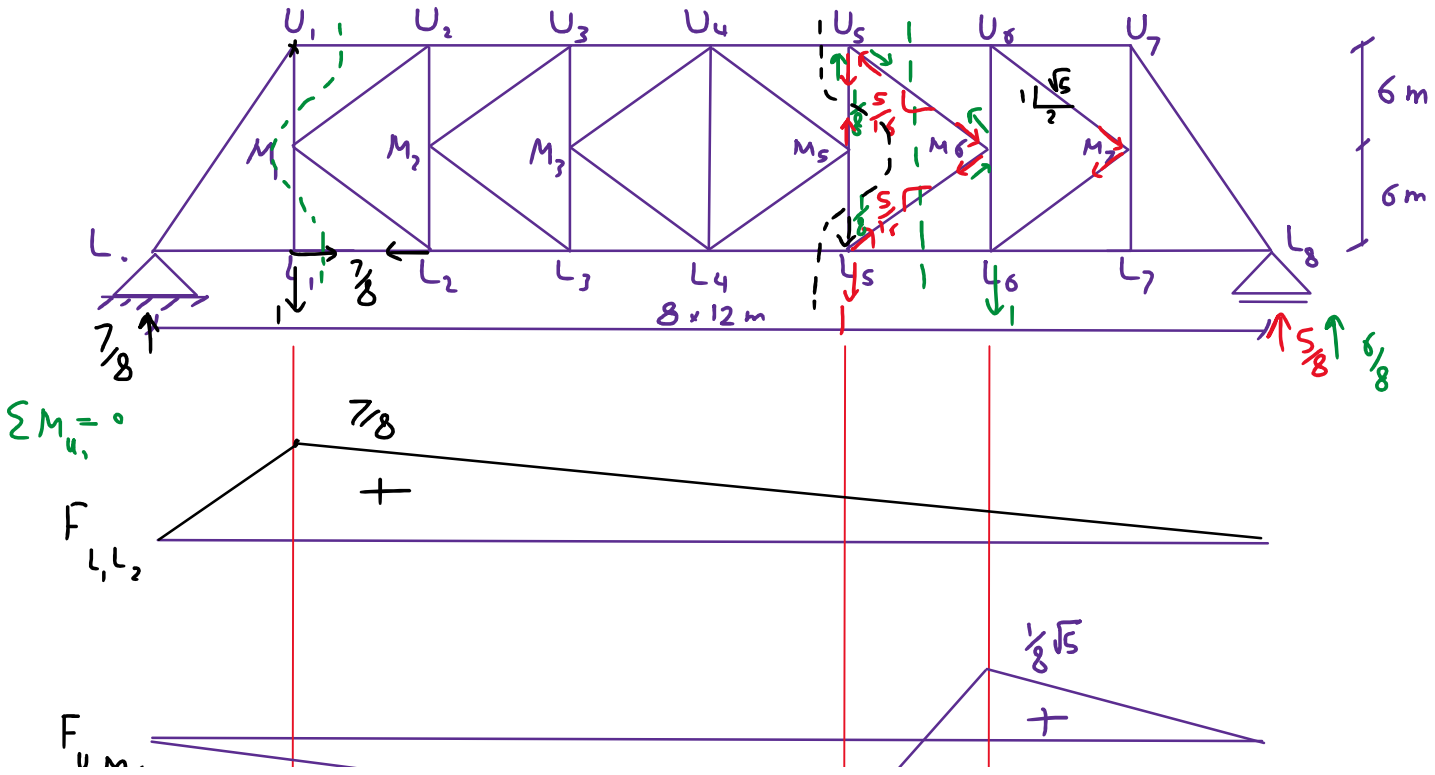


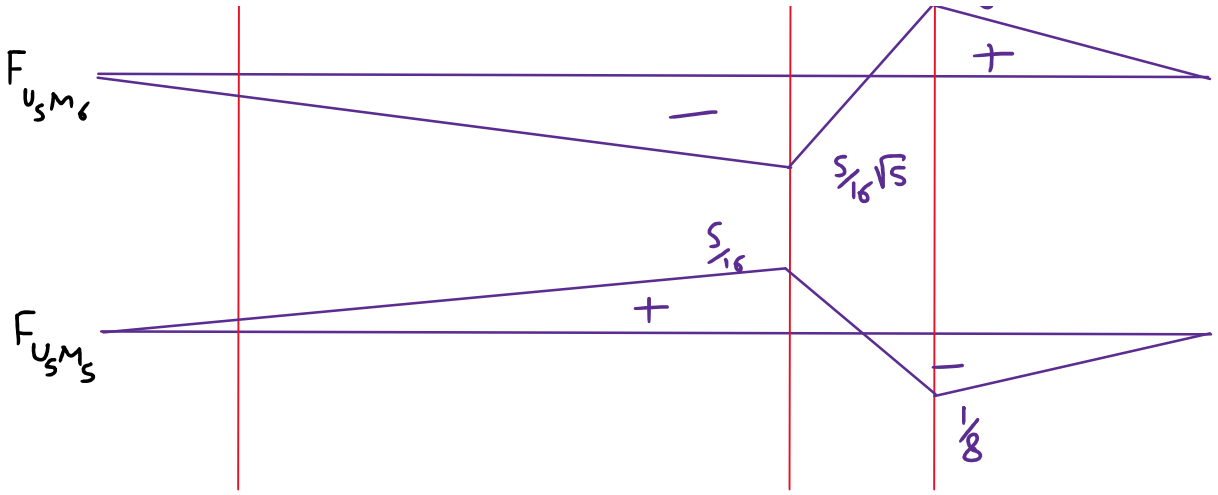

M_D^+
D
2 m
1
1
2
2
1
1
M_D^-
D
2 m
1
1
2
2
1
1

مثال: خط تأثیر نیروی اعضای L_1L_2، U_1L_1 و U_1L_2 را رسم نمایید.

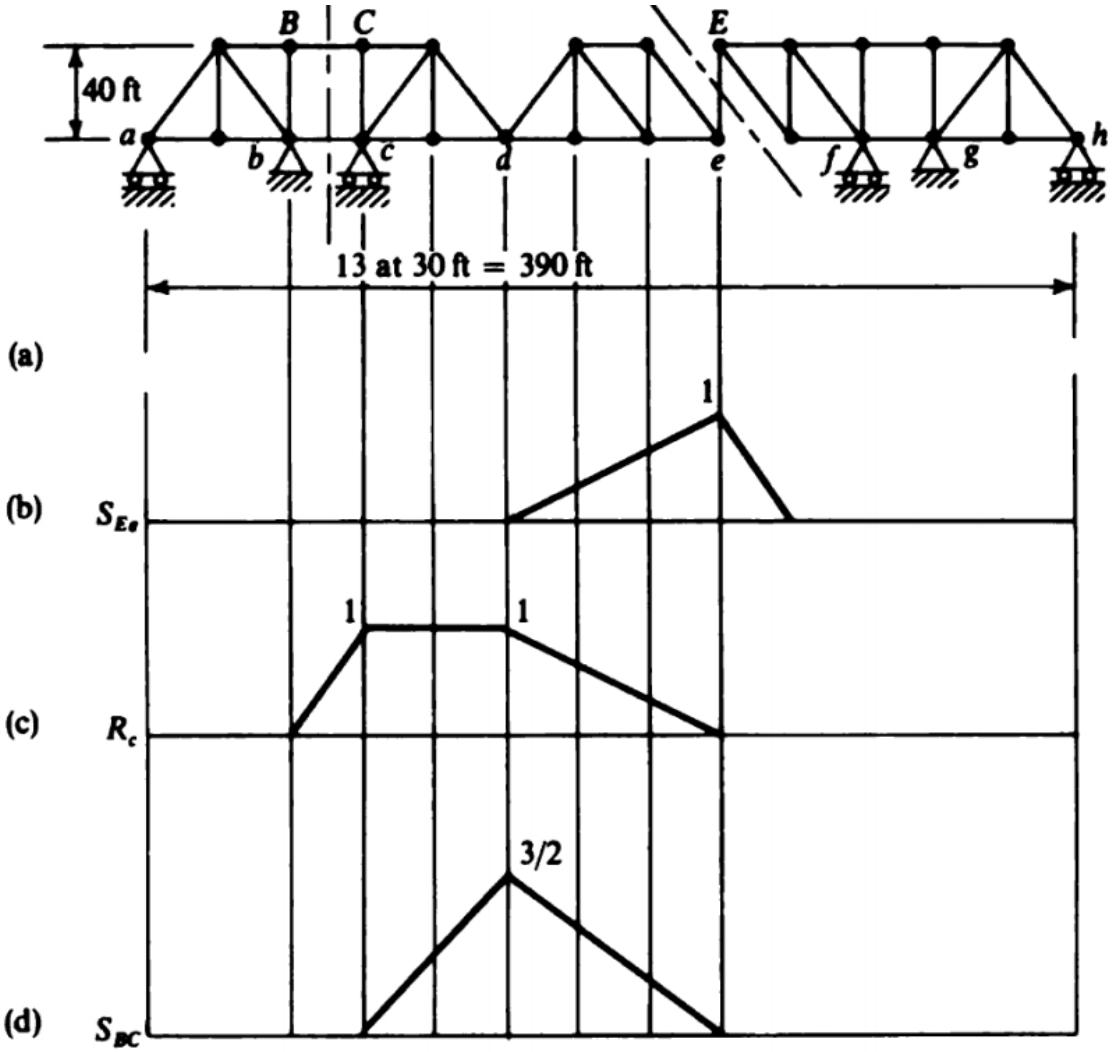


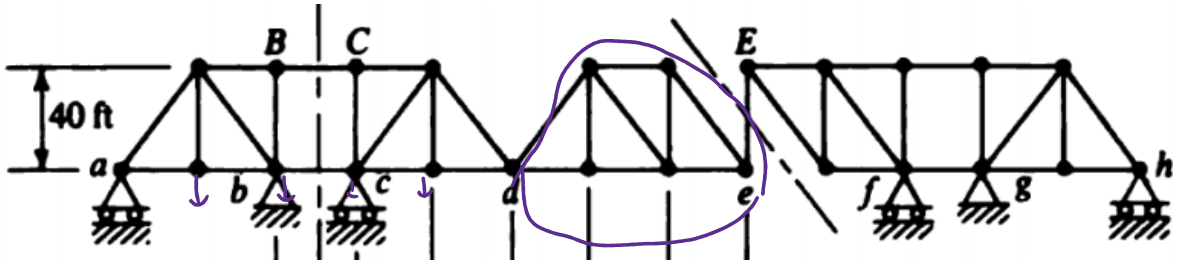
مثال: خط تأثیر نیروی اعضای L_1L_2، U_5M_6 و U_5M_5 را رسم نمایید.

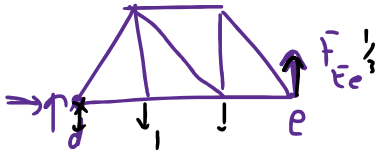


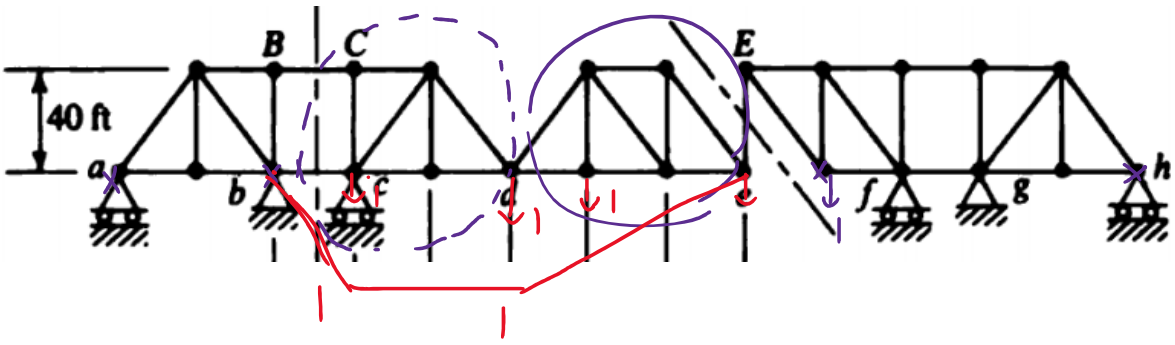


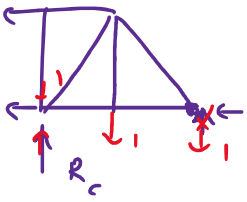
مثال: مطلوب ات خط تأثیر اعضاء Ee ، Bc وعکس العمل تکیه گاه c .



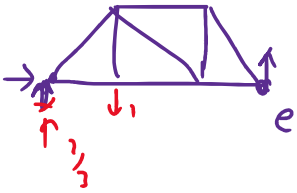

B
C
E
40 ft
a
b
c
d
e
f
g
h

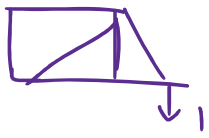

e

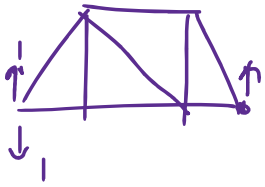

B
C
E
40 ft
a
b
c
d
e
f
g
h

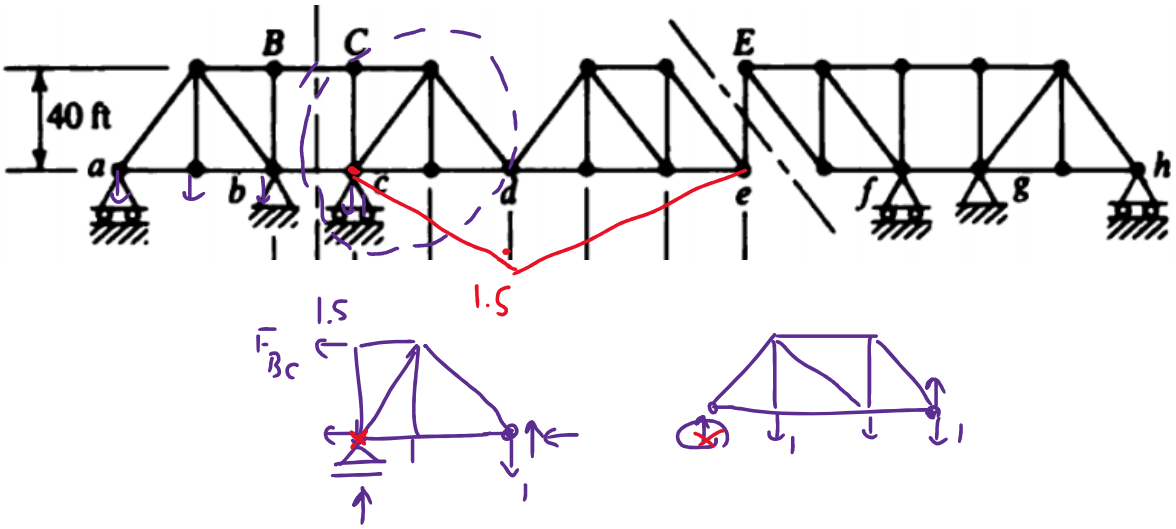


$\Sigma F_y = 0$

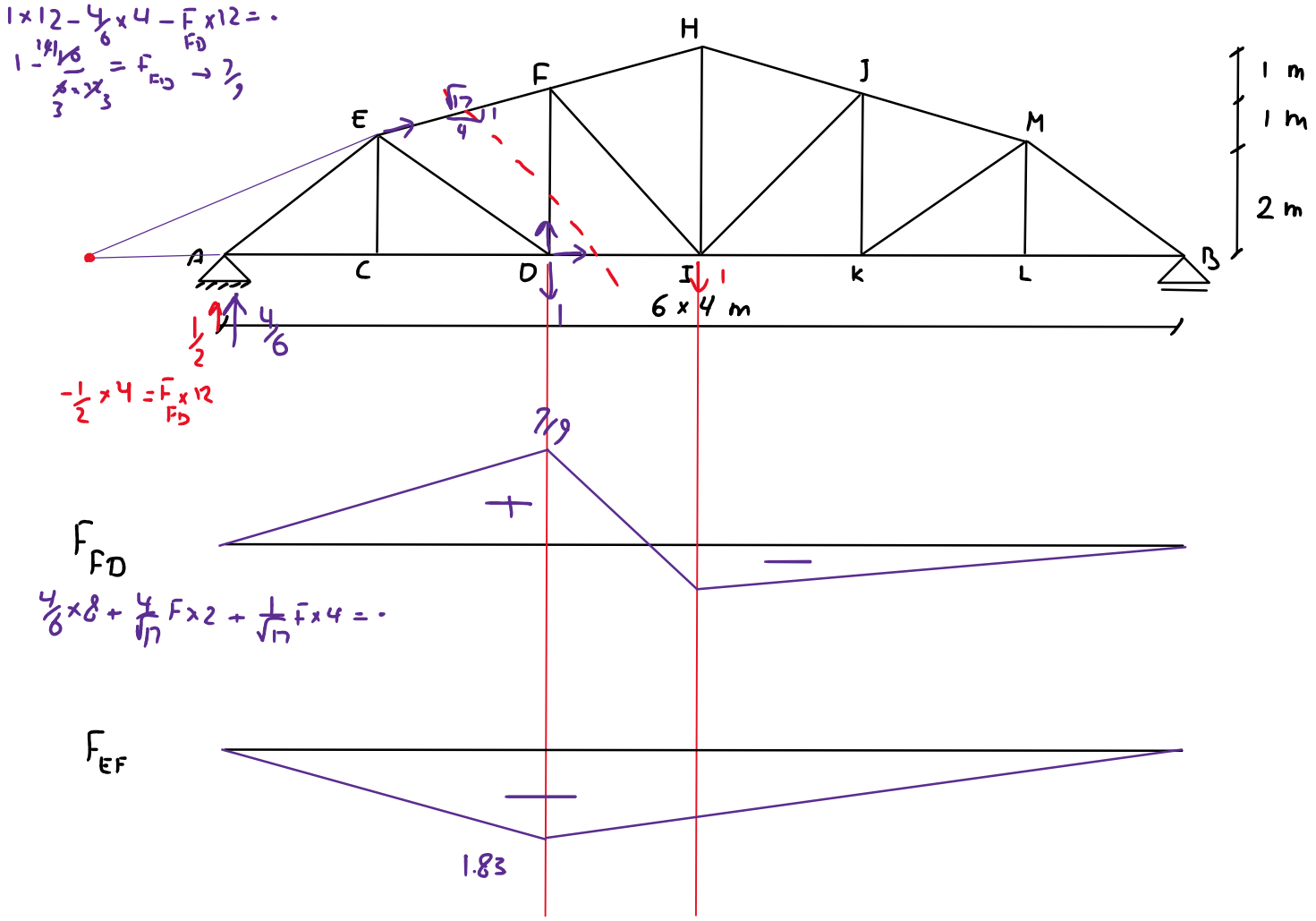

e



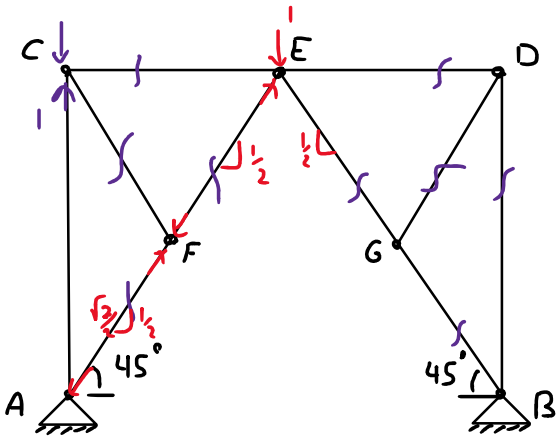



B
C
E
40 ft
a
b
c
d
e
f
g
h
1.5
1.5

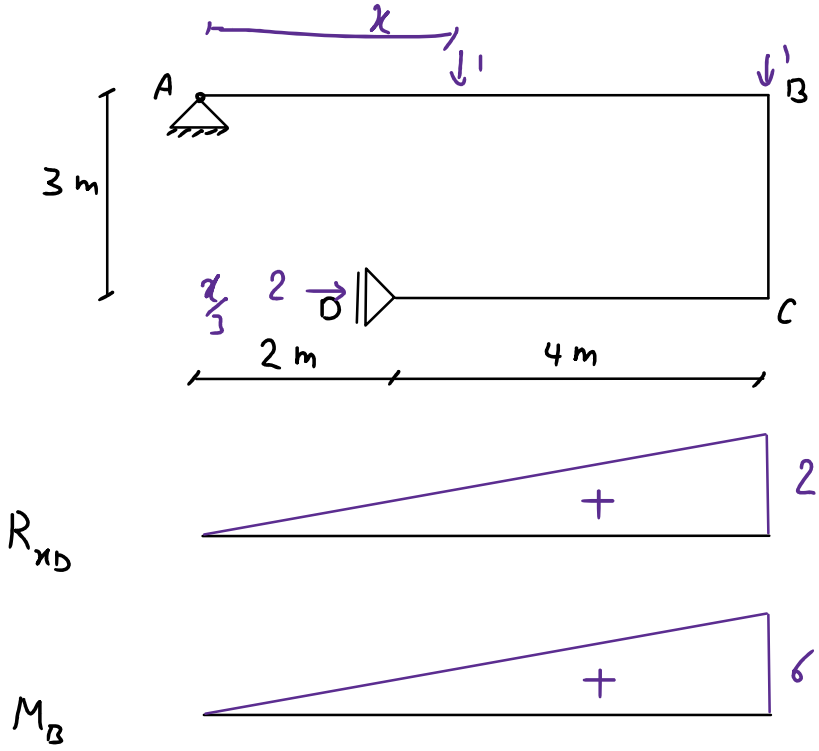
مثال: خط تأثیر اعضای FD و EF را رسم کنید.



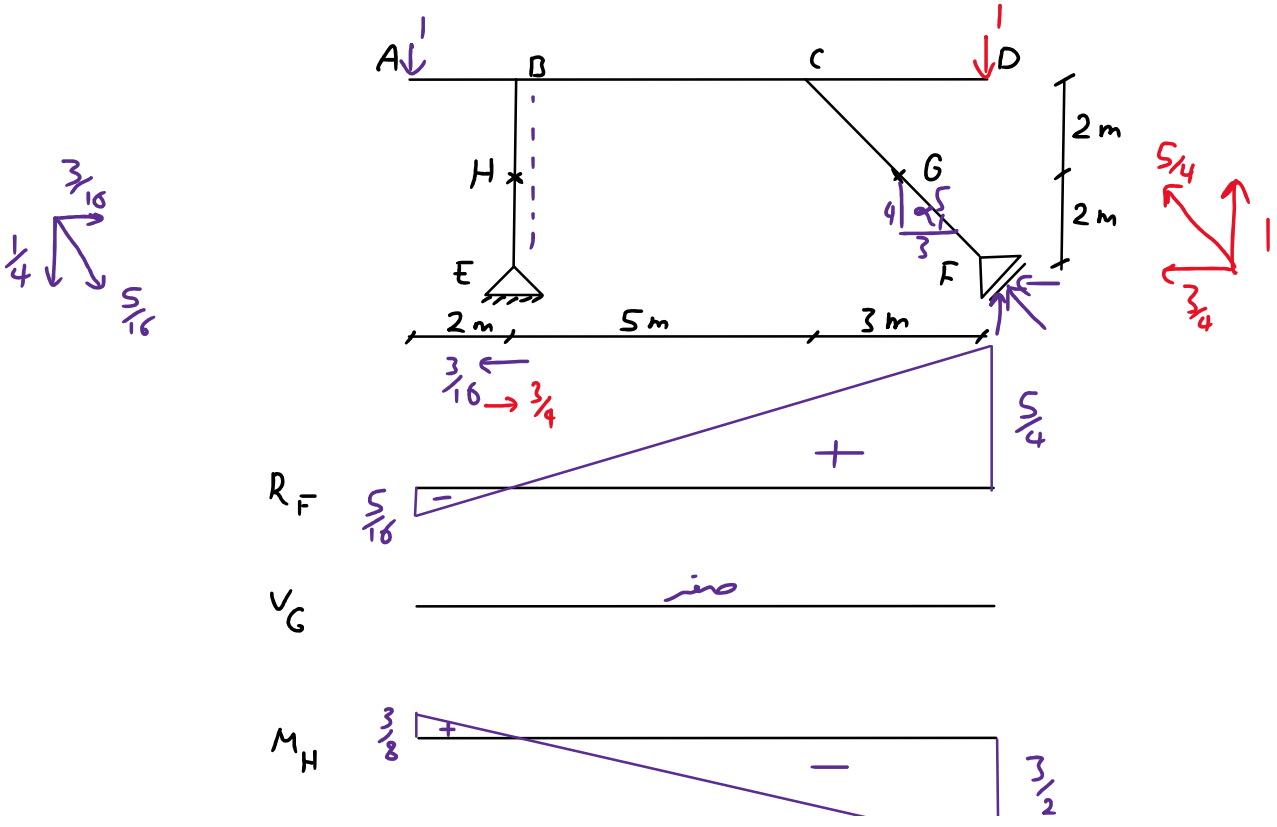
مثال: خط تأثیر اعضای AC و AF را رسم نمایید؛ وقتی بار روی CD حرکت می‌کند.

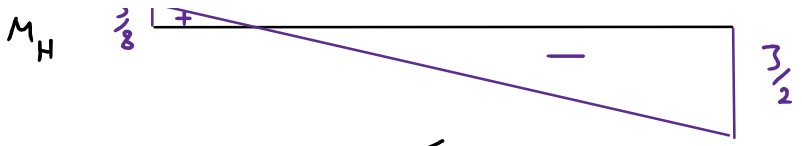


مثال: خطوط تأثیر R_{xD} و M_B وقتی بار روی عضو AB حرکت می‌کند.

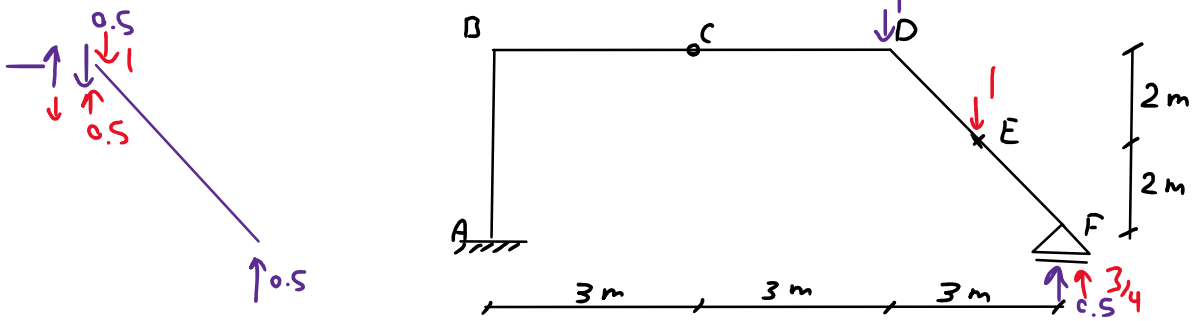


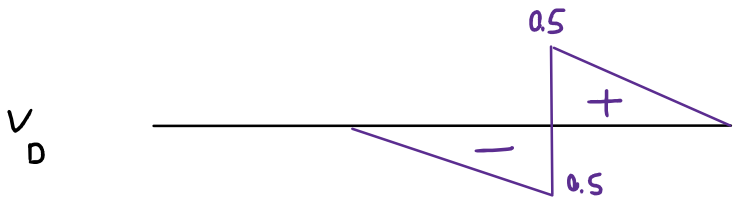
مثال: وقتی بار روی تیر حرکت می‌کند، خط تأثیر R_F، V_G و M_H را به دست آورید.

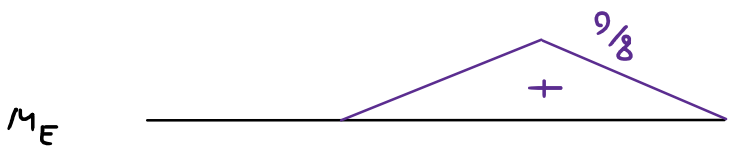




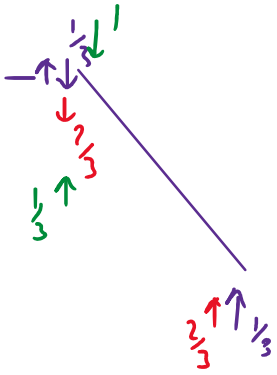
مثال: خط تأثیر V_D و M_E را وقتی بار روی BCDF حرکت می‌کند، رسم کنید.

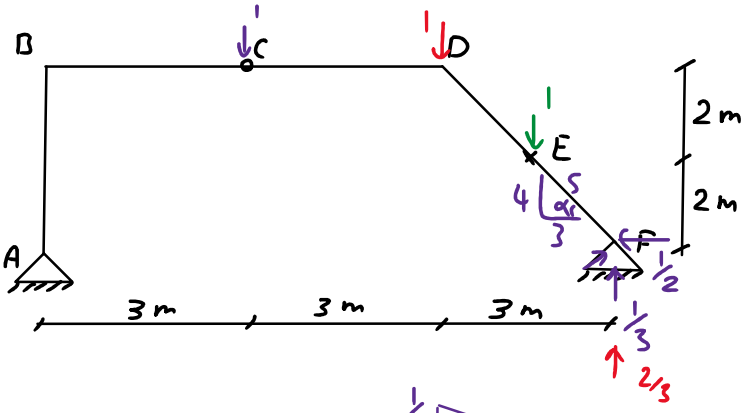


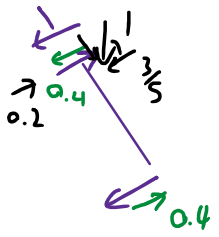


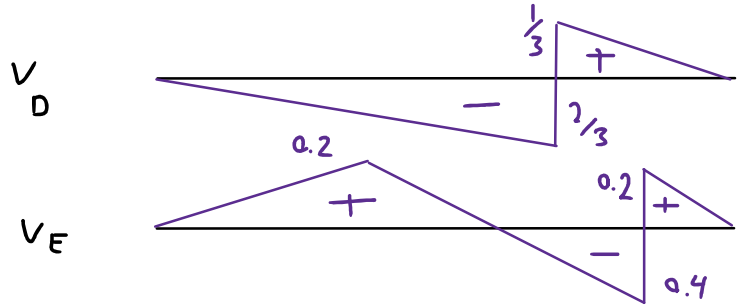


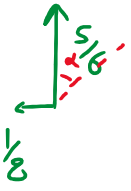
مثال: خط تأثیر V_D، V_E و M_E را وقتی بار روی BCDF حرکت می‌کند، رسم کنید.



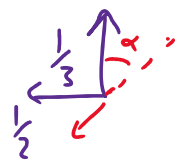






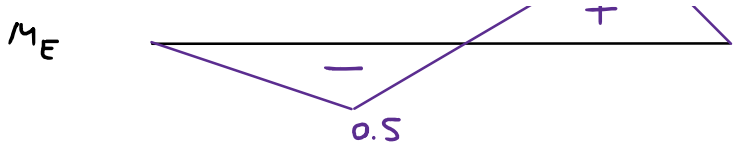


$\frac{5}{6}\left(\frac{3}{5}\right) - \frac{1}{8}\left(\frac{4}{5}\right) = 0.4$



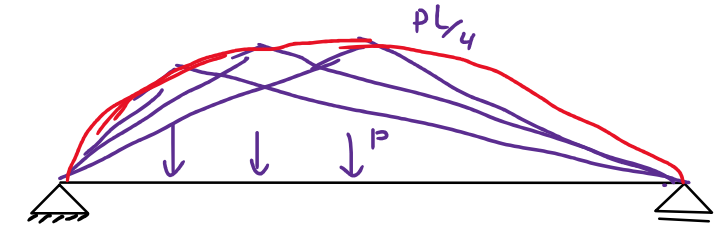
$\frac{1}{3}\cos\alpha - \frac{1}{2}\sin\alpha$

$\frac{1}{3}\left(\frac{3}{5}\right) - \frac{1}{2}\left(\frac{4}{5}\right) = -\frac{1}{5}$

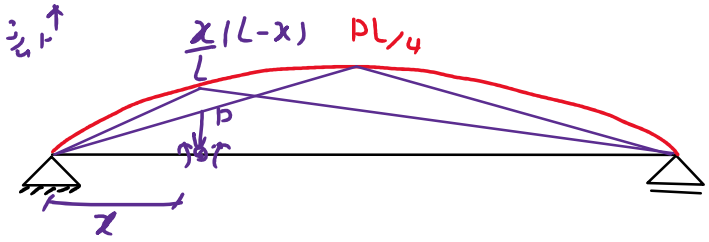

M_E
+
−
0.S

دیاگرام پوش (envelope)

① بار متمرکز در حال حرکت

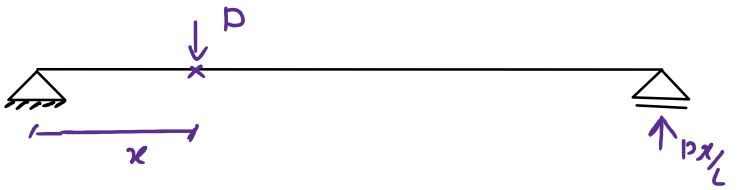


از خط تاثیر



تابع پوش لنگر $M = P\left(x - \frac{x^2}{L}\right)$

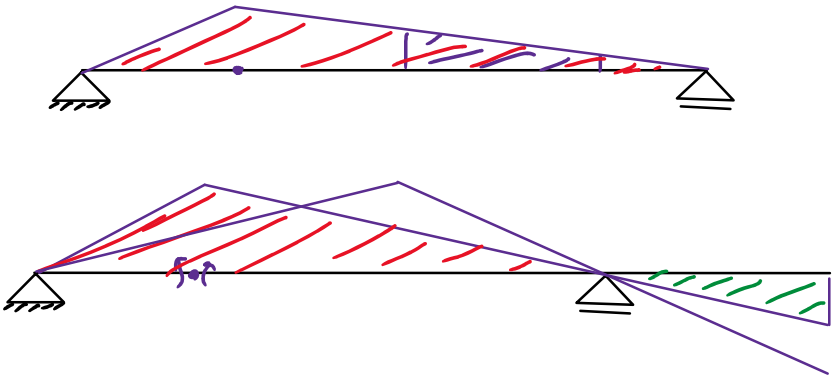
از تعلیل



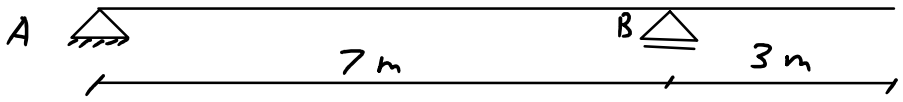
$M = \frac{Px}{L}(L-x)$

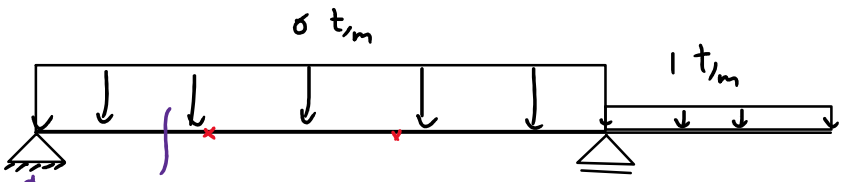
تابع پوش $M = P\left(x - \frac{x^2}{L}\right)$

② بار گسترده با طول متغیر

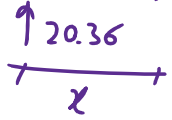


مثال: دیاگرام پوش لنگر برای تیر شکل زیر را به دست آورید. بار مرده ضریبدار برابر $1\ t/m$ و بار زنده ضریبدار برابر $5\ t/m$ می‌باشد.

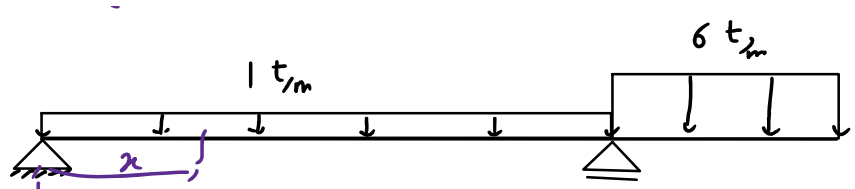




پوش لنگر مثبت

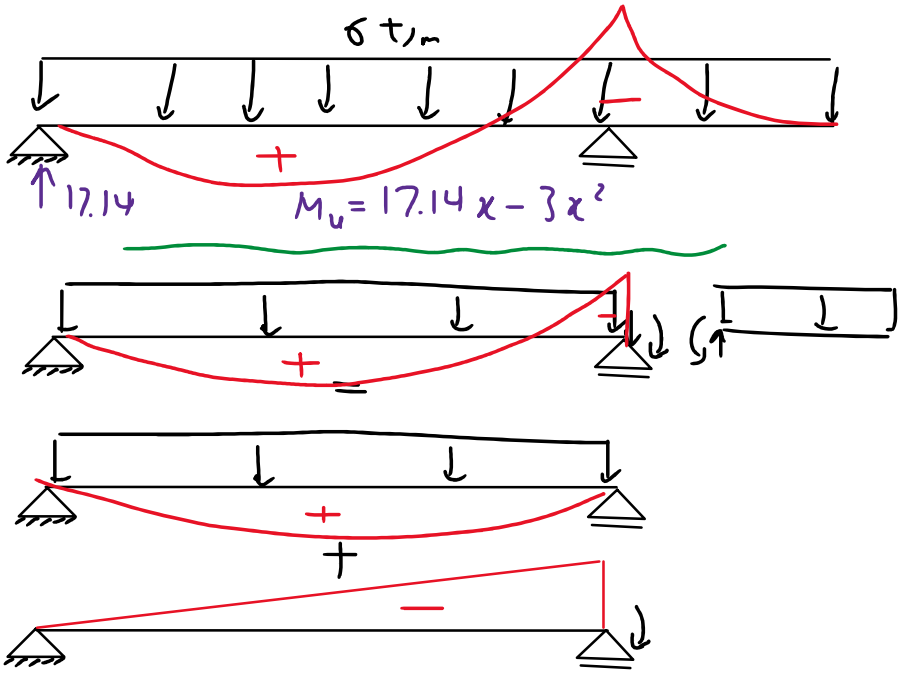


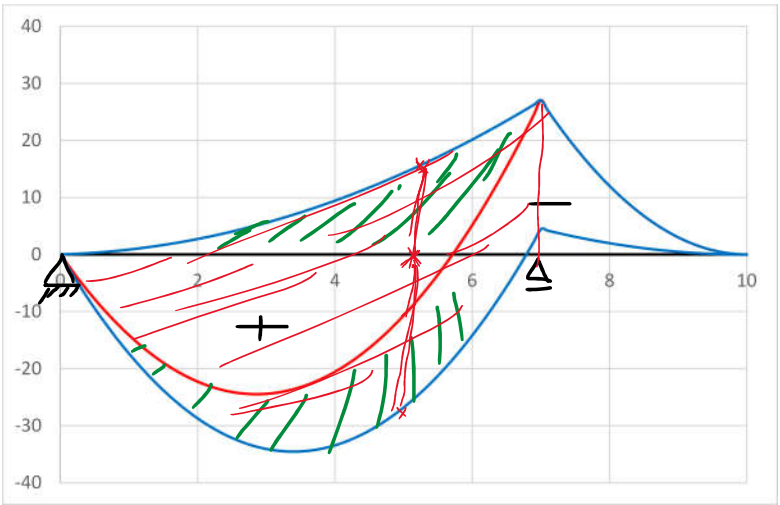
$M_u = 20.36x - 3x^2$



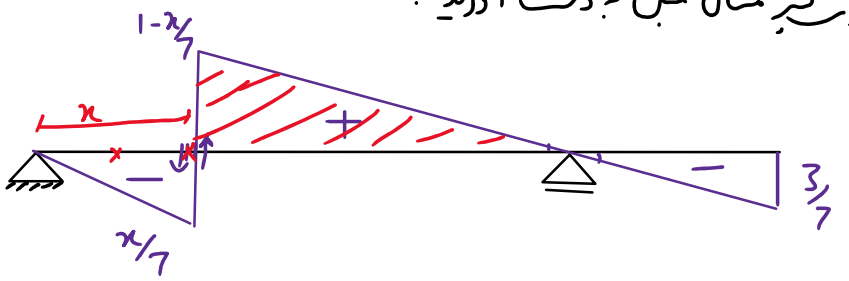
بوش لگرمن

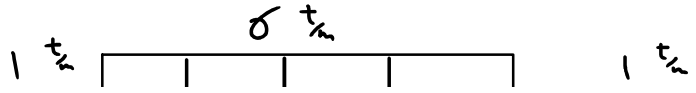
$M_u = -0.357x - 0.5x^2$

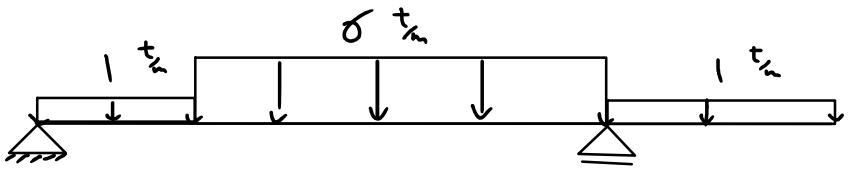




مثال : دیاگرام بدنی برش را برای تیر مثال قبل به دست آورید .



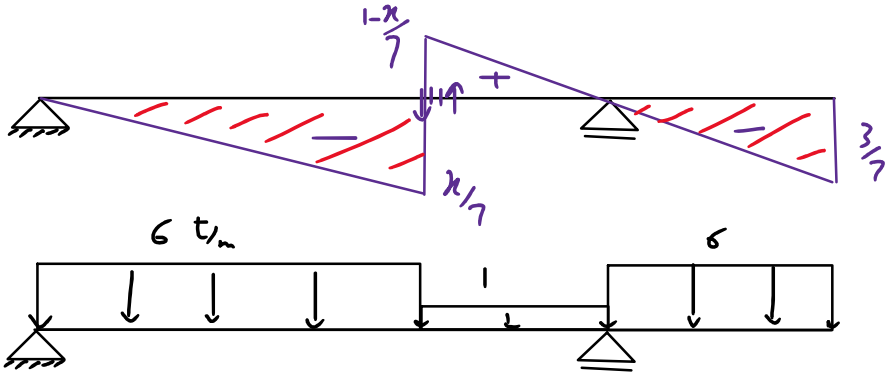




برش مثبت $V = -1\left(\frac{1}{2}\times\frac{x}{7}\times x\right) + 6\left(\frac{1}{2}\times\left(1-\frac{x}{7}\right)\times(7-x)\right) - 1\left(\frac{1}{2}\times\frac{3}{7}\times 3\right)$

$V = -\frac{1}{14}x^2 + \frac{3}{7}(49+x^2-14x) - \frac{9}{14}$

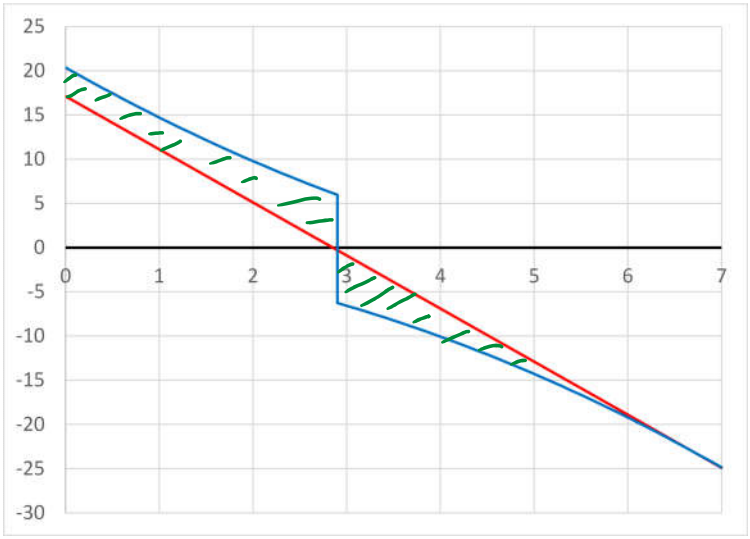
$$V = \frac{5}{14}x^2 - 6x + \frac{285}{14}$$



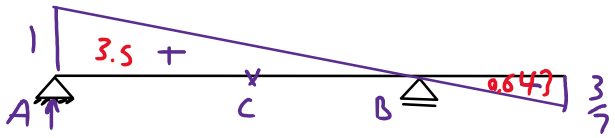
برش منفی $V = -6\left(\frac{1}{2}\times\frac{x}{7}\times x\right) + 1\left(\frac{1}{2}\times\left(1-\frac{x}{7}\right)\times(7-x)\right) - 6\left(\frac{1}{2}\times\frac{3}{7}\times 3\right)$

$V = -\frac{6}{14}x^2 + \frac{1}{14}(49+x^2-14x) - \frac{54}{14}$

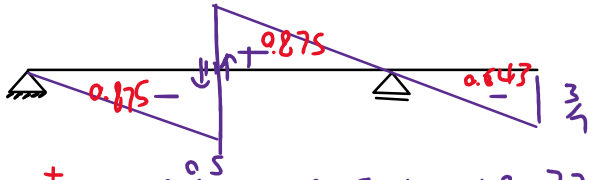
$$V = \frac{-5}{14}x^2 - x - \frac{5}{14}$$



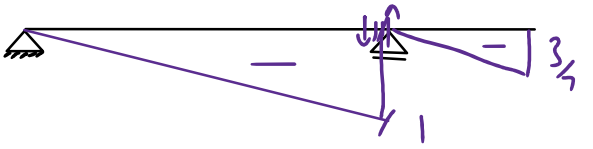
رسم بدنه به صورت تقریبی

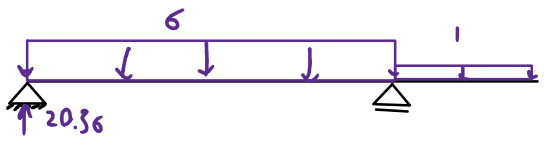


$V_A = +6 \times 3.5 - 1 \times 0.643 = 20.36$

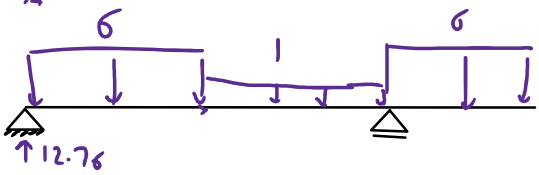


$V_C^+ = -1 \times 0.875 + 6 \times 0.875 - 1 \times 0.643 = 3.73$

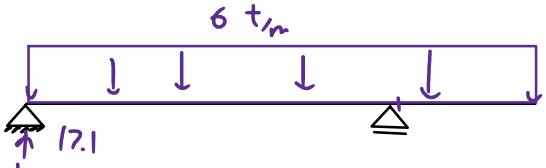




$V_A = 20.36$

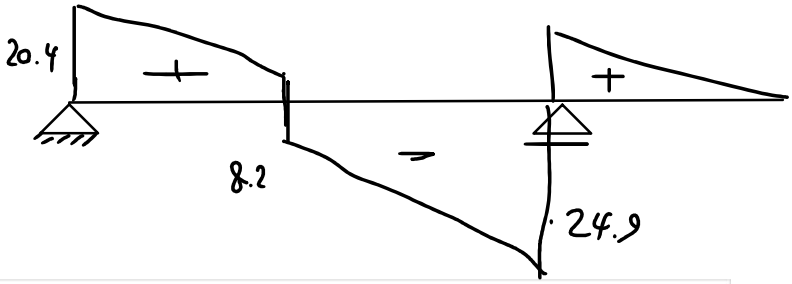


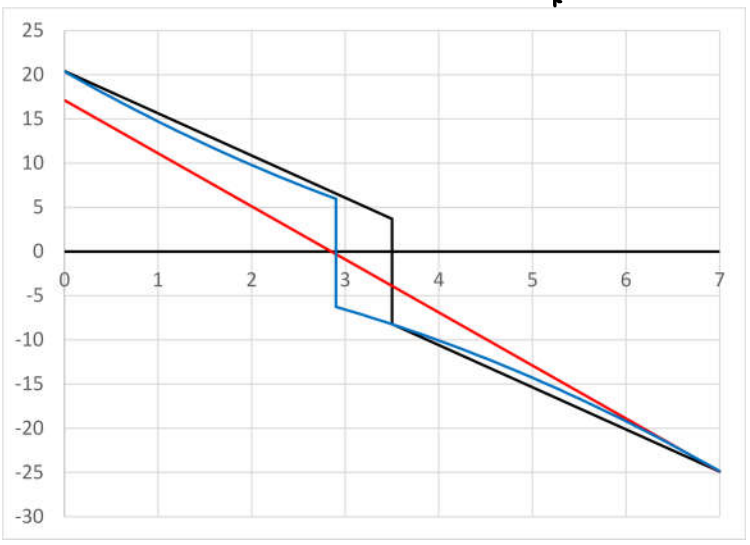
$V_C^- = 12.76 - 6 \times 3.5 = -8.23$

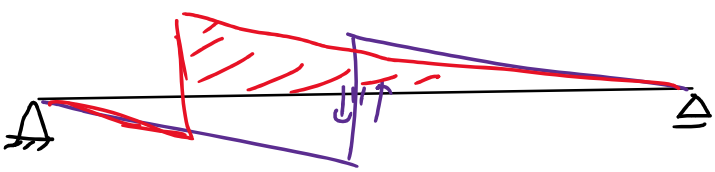


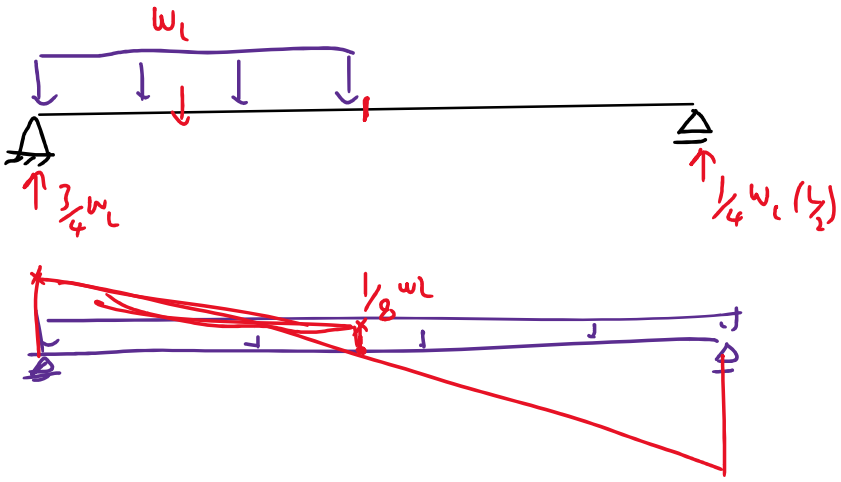
$V_{BL} = 17.1 - 6 \times 7 = -24.9$

$V_{BR} = 6 \times 3 = 18$








w_L
$\frac{3}{4}w_L$
$\frac{1}{4}w_L\left(\frac{L}{2}\right)$
$\frac{1}{8}wL$